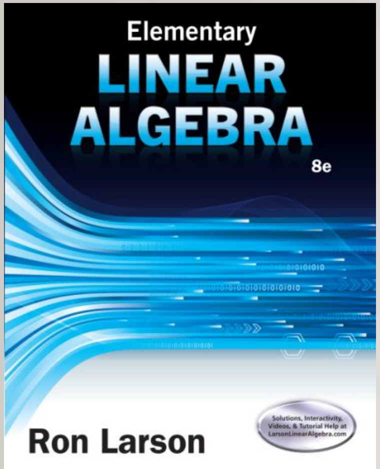
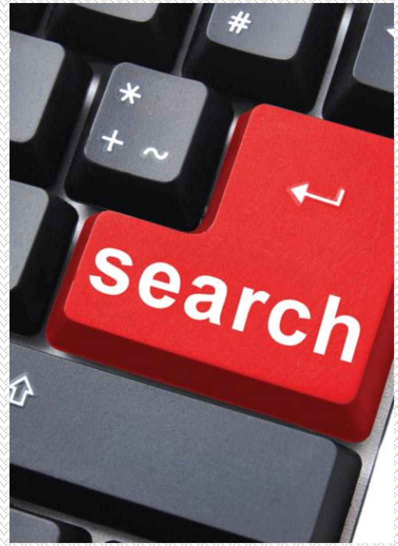


CHAPTER 2
MATRICES

2.1 Operations with Matrices
2.2 Properties of Matrix Operations
2.3 The Inverse of a Matrix
2.4 Elementary Matrices
2.5 Markov Chain
2.6 More Applications of Matrix Operations

CH 2 Linear Algebra Applied

Flight Crew Scheduling (p.47)



Information Retrieval (p.58)

Beam Deflection (p.64)

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Data Encryption (p.94)

2.1 Operations with Matrices

- Matrix:

$$A=[a_{ij}]_{m\times n}=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \in M_{m\times n}$$

(i, j)-th entry: a_{ij}

row: m

column: n

size: $m\times n$

- *i*-th row vector:

$$r_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$$

row matrix

- *j*-th column vector:

$$c_j = \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{mj} \end{bmatrix}$$

column matrix

- Square matrix: $m = n$

- Diagonal matrix:

$$A = diag(d_1, d_2, \cdots, d_n) = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \in M_{n\times n}$$

- Trace:

If $A = [a_{ij}]_{n\times n}$

Then $Tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$

- Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$\Rightarrow \quad r_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}$$

$$\Rightarrow \quad c_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \; c_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

- Equal matrix:

If $A = [a_{ij}]_{m\times n}, \; B = [b_{ij}]_{m\times n}$

Then $A = B$ if and only if $a_{ij} = b_{ij} \;\; \forall\, 1 \le i \le m,\; 1 \le j \le n$

- Ex 1: (Equal matrix)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If $A = B$

Then $a = 1,\;\; b = 2,\;\; c = 3,\;\; d = 4$

- Matrix addition:

$$\text{If } A=[a_{ij}]_{m\times n},\ B=[b_{ij}]_{m\times n}$$

$$\text{Then } A+B=[a_{ij}]_{m\times n}+[b_{ij}]_{m\times n}=[a_{ij}+b_{ij}]_{m\times n}$$

- Ex 2: (Matrix addition)

$$\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}=\begin{bmatrix} -1+1 & 2+3 \\ 0-1 & 1+2 \end{bmatrix}=\begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}+\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}=\begin{bmatrix} 1-1 \\ -3+3 \\ -2+2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Scalar multiplication:

If $A=[a_{ij}]_{m\times n}$, c : scalar

Then $cA=[ca_{ij}]_{m\times n}$

- Matrix subtraction:

$A-B=A+(-1)B$

- Ex 3: (Scalar multiplication and matrix subtraction)

$$A=\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \qquad B=\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

Find (a) $3A$, (b) $-B$, (c) $3A-B$

Sol:

(a)

$$3A=3\begin{bmatrix}1&2&4\\-3&0&-1\\2&1&2\end{bmatrix}=\begin{bmatrix}3(1)&3(2)&3(4)\\3(-3)&3(0)&3(-1)\\3(2)&3(1)&3(2)\end{bmatrix}=\begin{bmatrix}3&6&12\\-9&0&-3\\6&3&6\end{bmatrix}$$

(b)

$$-B=(-1)\begin{bmatrix}2&0&0\\1&-4&3\\-1&3&2\end{bmatrix}=\begin{bmatrix}-2&0&0\\-1&4&-3\\1&-3&-2\end{bmatrix}$$

(c)

$$3A-B=\begin{bmatrix}3&6&12\\-9&0&-3\\6&3&6\end{bmatrix}-\begin{bmatrix}2&0&0\\1&-4&3\\-1&3&2\end{bmatrix}=\begin{bmatrix}1&6&12\\-10&4&-6\\7&0&4\end{bmatrix}$$

- Matrix multiplication:

If $A = [a_{ij}]_{m\times n}, \; B = [b_{ij}]_{n\times p}$

Then $AB = [a_{ij}]_{m\times n}[b_{ij}]_{n\times p} = [c_{ij}]_{m\times p}$

Size of AB

where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & \vdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{in} \\ \\ \end{bmatrix}$$

- Notes: (1) $A+B = B+A$, (2) $AB \neq BA$

- Ex 4: (Find AB)

$$A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$$

Sol:

$$AB = \begin{bmatrix} (-1)(-3)+(3)(-4) & (-1)(2)+(3)(1) \\ (4)(-3)+(-2)(-4) & (4)(2)+(-2)(1) \\ (5)(-3)+(0)(-4) & (5)(2)+(0)(1) \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}$$

- Matrix form of a system of linear equations:

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right.$$

m linear equations

$$\Downarrow$$

$$\underset{\displaystyle A}{\underset{\shortparallel}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}} \underset{\displaystyle x}{\underset{\shortparallel}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}} = \underset{\displaystyle b}{\underset{\shortparallel}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}}$$

$$\downarrow$$

Single matrix equation

$$\underset{m\times n}{A}\,\underset{n\times 1}{x} = \underset{m\times 1}{b}$$

- Partitioned matrices:

submatrix

$$A=\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A=\left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]=\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$$A=\left[\begin{array}{c|c|c|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]=\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}$$

- Linear combination of column vectors:

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}=\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \qquad x=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\Rightarrow Ax=\begin{bmatrix} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n \\ \vdots \\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n \end{bmatrix}_{m\times 1}=x_1\underbrace{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}}_{c_1}+x_2\underbrace{\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}}_{c_2}+\cdots+x_n\underbrace{\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}}_{c_n}$$

- Ex 7: (Solve a system of linear equations)

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 0 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & 3 \\ 7x_1 & + & 7x_2 & + & 8x_3 & = & 6 \end{array}$$

(infinitely many solutions)

$$\Rightarrow A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}, c_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, c_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, c_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

$$\Rightarrow Ax = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 4x_1 + 5x_2 + 6x_3 \\ 7x_1 + 8x_2 + 9x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} = b$$

$$\Rightarrow 1\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + (-1)\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} \quad \text{(one solution : } x = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \text{ i.e. } x_1 = 1, x_2 = 1, x_3 = -1)$$

Key Learning in Section 2.1

- Determine whether two matrices are equal.
- Add and subtract matrices and multiply a matrix by a scalar.
- Multiply two matrices.
- Use matrices to solve a system of linear equations.
- Partition a matrix and write a linear combination of column vectors.

Keywords in Section 2.1

- row vector: 列向量
- column vector: 行向量
- diagonal matrix: 對角矩陣
- trace: 跡數
- equality of matrices: 相等矩陣
- matrix addition: 矩陣相加
- scalar multiplication: 純量乘法(純量積)
- matrix subtraction: 矩陣相減
- matrix multiplication: 矩陣乘法
- partitioned matrix: 分割矩陣
- linear combination: 線性組合

2.2 Properties of Matrix Operations

- Three basic matrix operators:

  (1) matrix addition

  (2) scalar multiplication

  (3) matrix multiplication

- Zero matrix: $0_{m\times n}$

- Identity matrix of order *n*: I_n

- Properties of matrix addition and scalar multiplication:

If $A, B, C \in M_{m\times n}$, c, d : scalar

Then (1) $A+B = B + A$

(2) $A + (B + C) = (A + B) + C$

(3) $(cd) A = c (dA)$

(4) $1A = A$

(5) $c(A+B) = cA + cB$

(6) $(c+d) A = cA + dA$

- Properties of zero matrices:

If $A \in M_{m\times n}, \quad c: scalar$

Then (1) $A + 0_{m\times n} = A$

(2) $A + (-A) = 0_{m\times n}$

(3) $cA = 0_{m\times n} \Rightarrow c = 0 \; or \; A = 0_{m\times n}$

- Notes:

(1) $0_{m\times n}$: **the additive identity** for the set of all $m\times n$ matrices

(2) $-A$: **the additive inverse** of A

- Properties of matrix multiplication:

$$(1)\ A(BC) = (AB)C$$

$$(2)\ A(B+C) = AB + AC$$

$$(3)\ (A+B)C = AC + BC$$

$$(4)\ c(AB) = (cA)B = A(cB)$$

- Properties of identity matrix:

If $A \in M_{m\times n}$

Then (1) $AI_n = A$

(2) $I_m A = A$

- Transpose of a matrix:

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in M_{m\times n}$$

$$\text{Then } A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \in M_{n\times m}$$

- Ex 8: (Find the transpose of the following matrix)

$$(a)\quad A=\begin{bmatrix}2\\8\end{bmatrix}\qquad (b)\quad A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\qquad (c)\quad A=\begin{bmatrix}0&1\\2&4\\1&-1\end{bmatrix}$$

Sol: (a)

$$A=\begin{bmatrix}2\\8\end{bmatrix}\quad\Rightarrow A^T=\begin{bmatrix}2&8\end{bmatrix}$$

(b)

$$A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\Rightarrow A^T=\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}$$

(c)

$$A=\begin{bmatrix}0&1\\2&4\\1&-1\end{bmatrix}\quad\Rightarrow A^T=\begin{bmatrix}0&2&1\\1&4&-1\end{bmatrix}$$

- Properties of transposes:

$$(1)\ \ (A^T)^T = A$$

$$(2)\ \ (A+B)^T = A^T + B^T$$

$$(3)\ \ (cA)^T = c(A^T)$$

$$(4)\ \ (AB)^T = B^T A^T$$

- Symmetric matrix:

  A square matrix A is **symmetric** if $A = A^T$

- Skew-symmetric matrix:

  A square matrix A is **skew-symmetric** if $A^T = -A$

- Ex:

  If $A = \begin{bmatrix} 1 & 2 & 3 \\ a & 4 & 5 \\ b & c & 6 \end{bmatrix}$ is symmetric, find a, b, c?

Sol:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ a & 4 & 5 \\ b & c & 6 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & a & b \\ 2 & 4 & c \\ 3 & 5 & 6 \end{bmatrix}$$

$$A = A^T$$
$$\Rightarrow a = 2, b = 3, c = 5$$

- Ex:

If $A=\begin{bmatrix}0 & 1 & 2\\ a & 0 & 3\\ b & c & 0\end{bmatrix}$ is a skew-symmetric, find *a*, *b*, *c*?

Sol:

$$A=\begin{bmatrix}0 & 1 & 2\\ a & 0 & 3\\ b & c & 0\end{bmatrix} \qquad -A^T=\begin{bmatrix}0 & -a & -b\\ -1 & 0 & -c\\ -2 & -3 & 0\end{bmatrix}$$

$$A=-A^T \;\Rightarrow\; a=-1, b=-2, c=-3$$

- Note: AA^T is symmetric

Pf: $(AA^T)^T=(A^T)^T A^T = AA^T$

$\therefore AA^T$ is symmetric

- Real number:

$ab = ba$ (Commutative law for multiplication)

- Matrix:

$$\underset{m\times n}{A}\underset{n\times p}{B} \neq BA$$

Three situations:

(1) If $m \neq p$, then AB is defined , BA is undefined.

(2) If $m = p, m \neq n$, then $AB \in M_{m\times m}$, $BA \in M_{n\times n}$ (Sizes are not the same)

(3) If $m = p = n$, then $AB \in M_{m\times m}$, $BA \in M_{m\times m}$

(Sizes are the same, but matrices are not equal)

- Ex 4:

Sow that AB and BA are not equal for the matrices.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$$

Sol:

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 4 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 4 & -2 \end{bmatrix}$$

- Note: $AB \neq BA$

- Real number:

$$ac = bc,\ \ c \neq 0$$

$$\Rightarrow\ \ a = b \qquad \text{(Cancellation law)}$$

- Matrix:

$$AC = BC \qquad C \neq 0$$

(1) If C is invertible, then $A = B$

(2) If C is not invertible, then $A \neq B$ (Cancellation is not valid)

- Ex 5: (An example in which cancellation is not valid)

Show that $AC=BC$

$$A=\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix},\quad B=\begin{bmatrix}2 & 4\\ 2 & 3\end{bmatrix},\quad C=\begin{bmatrix}1 & -2\\ -1 & 2\end{bmatrix}$$

Sol:

$$AC=\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & -2\\ -1 & 2\end{bmatrix}=\begin{bmatrix}-2 & 4\\ -1 & 2\end{bmatrix}$$

$$BC=\begin{bmatrix}2 & 4\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & -2\\ -1 & 2\end{bmatrix}=\begin{bmatrix}-2 & 4\\ -1 & 2\end{bmatrix}$$

So $\quad AC=BC$

But $\quad A\neq B$

Key Learning in Section 2.2

- Use the properties of matrix addition, scalar multiplication, and zero matrices.
- Use the properties of matrix multiplication and the identity matrix.
- Find the transpose of a matrix.

Keywords in Section 2.2

- zero matrix: 零矩陣
- identity matrix: 單位矩陣
- transpose matrix: 轉置矩陣
- symmetric matrix: 對稱矩陣
- skew-symmetric matrix: 反對稱矩陣

2.3 The Inverse of a Matrix

- Inverse matrix:

Consider $A \in M_{n\times n}$

If there exists a matrix $B \in M_{n\times n}$ such that $AB = BA = I_n$,

Then (1) A is **invertible** (or **nonsingular**)

(2) B is **the inverse** of A

- Note:

A matrix that does not have an inverse is called **noninvertible** (or **singular**).

- Thm 2.7: (The inverse of a matrix is unique)

If B and C are both inverses of the matrix A, then $B = C$.

Pf:

$$AB = I$$
$$C(AB) = CI$$
$$(CA)B = C$$
$$IB = C$$
$$B = C$$

Consequently, the inverse of a matrix is unique.

- Notes:

(1) The inverse of A is denoted by A^{-1}

(2) $AA^{-1} = A^{-1}A = I$

- Find the inverse of a matrix by Gauss-Jordan Elimination:

$$[A \mid I] \xrightarrow{\text{Gauss-Jordan Elimination}} [I \mid A^{-1}]$$

- Ex 2: (Find the inverse of the matrix)

$$A=\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$$

Sol:

$$AX = I$$

$$\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_{11}+4x_{21} & x_{12}+4x_{22} \\ -x_{11}-3x_{21} & -x_{12}-3x_{22} \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{array}{rcrcr} x_{11} & + & 4x_{21} & = & 1 \\ -x_{11} & - & 3x_{21} & = & 0 \end{array} \quad (1)$$

$$\begin{array}{rcrcr} x_{12} & + & 4x_{22} & = & 0 \\ -x_{12} & - & 3x_{22} & = & 1 \end{array} \quad (2)$$

$$(1) \Rightarrow \left[\begin{array}{rr:r} 1 & 4 & 1 \\ -1 & -3 & 0 \end{array}\right] \xrightarrow{r_{12}^{(1)},\ r_{21}^{(-4)}} \left[\begin{array}{rr:r} 1 & 0 & -3 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow x_{11} = -3,\ x_{21} = 1$$

$$(2) \Rightarrow \left[\begin{array}{rr:r} 1 & 4 & 0 \\ -1 & -3 & 1 \end{array}\right] \xrightarrow{r_{12}^{(1)},\ r_{21}^{(-4)}} \left[\begin{array}{rr:r} 1 & 0 & -4 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow x_{12} = -4,\ x_{22} = 1$$

Thus

$$X = A^{-1} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \ (AX = I = AA^{-1})$$

- Note:

$$\left[\begin{array}{rr:rr} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right] \xrightarrow[r_{12}^{(1)},\, r_{21}^{(-4)}]{\text{Gauss–Jordan Elimination}} \left[\begin{array}{rr:rr} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$A \qquad I \qquad\qquad\qquad I \qquad A^{-1}$$

If A can't be row reduced to I, then A is singular.

- Ex 3: (Find the inverse of the following matrix)

$$A=\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

Sol:

$$[A \;\vdots\; I]=\left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_{12}^{(-1)}}\left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right]\xrightarrow{r_{13}^{(6)}}\left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_{23}^{(4)}}\left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right]\xrightarrow{r_{3}^{(-1)}}\left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

$$\xrightarrow{r_{32}^{(1)}} \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \xrightarrow{r_{21}^{(1)}} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -1 & -4 & -1 \end{array}\right]$$

$$= [\, I \,\vdots\, A^{-1} \,]$$

So the matrix A is invertible, and its inverse is

$$A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$$

- Check:

$$AA^{-1} = A^{-1}A = I$$

- Power of a square matrix:

$$(1) A^0 = I$$

$$(2) A^k = \underbrace{AA\cdots A}_{k \text{ factors}} \qquad (k > 0)$$

$$(3) A^r \cdot A^s = A^{r+s} \qquad r, s : \text{integers}$$

$$(A^r)^s = A^{rs}$$

$$(4) D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

- Thm 2.8: (Properties of inverse matrices)

If A is an invertible matrix, k is a positive integer, and c is a scalar not equal to zero, then

(1) A^{-1} is invertible and $(A^{-1})^{-1} = A$

(2) A^k is invertible and $(A^k)^{-1} = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{k \text{ factors}} = (A^{-1})^k = A^{-k}$

(3) cA is invertible and $(cA)^{-1} = \dfrac{1}{c}A^{-1}, c \neq 0$

(4) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

- Thm 2.9: (The inverse of a product)

If A and B are invertible matrices of size n, then AB is invertible and

$$(AB)^{-1}=B^{-1}A^{-1}$$

Pf:

$$(AB)(B^{-1}A^{-1})=A(BB^{-1})A^{-1}=A(I)A^{-1}=(AI)A^{-1}=AA^{-1}=I$$

$$(B^{-1}A^{-1})(AB)=B^{-1}(A^{-1}A)B=B^{-1}(I)B=B^{-1}(IB)=B^{-1}B=I$$

If AB is invertible, then its inverse is unique.

So $(AB)^{-1}=B^{-1}A^{-1}$

- Note:

$$(A_1A_2A_3\cdots A_n)^{-1}=A_n^{-1}\cdots A_3^{-1}A_2^{-1}A_1^{-1}$$

- Thm 2.10: (Cancellation properties)

If C is an invertible matrix, then the following properties hold:

(1) If $AC=BC$, then $A=B$ (Right cancellation property)

(2) If $CA=CB$, then $A=B$ (Left cancellation property)

Pf:

$$AC = BC$$

$$(AC)C^{-1} = (BC)C^{-1} \qquad \text{(C is invertible, so } C^{-1} \text{ exists)}$$

$$A(CC^{-1}) = B(CC^{-1})$$

$$AI = BI$$

$$A = B$$

- Note:

If C is not invertible, then cancellation is not valid.

- Thm 2.11: (Systems of equations with unique solutions)

  If A is an invertible matrix, then the system of linear equations $Ax = b$ has a unique solution given by

$$x = A^{-1}b$$

Pf:

$$Ax = b$$

$$A^{-1}Ax = A^{-1}b \qquad (A \text{ is nonsingular})$$

$$Ix = A^{-1}b$$

$$x = A^{-1}b$$

If x_1 and x_2 were two solutions of equation $Ax = b.$

then $Ax_1 = b = Ax_2 \Rightarrow x_1 = x_2$ (Left cancellation property)

This solution is unique.

- Note:

For square systems (those having the same number of equations as variables), Theorem 2.11 can be used to determine whether the system has a unique solution.

- Note:

$Ax = b$ (A is an invertible matrix)

$$\left[A \mid b\right] \xrightarrow{A^{-1}} \left[A^{-1}A \mid A^{-1}b\right] = \left[I \mid A^{-1}b\right]$$

$$\left[A \mid b_1 \mid b_2 \mid \cdots \mid b_n\right] \xrightarrow{A^{-1}} \left[I \mid A^{-1}b_1 \mid \cdots \mid A^{-1}b_n\right]$$

Key Learning in Section 2.3

- Find the inverse of a matrix (if it exists).
- Use properties of inverse matrices.
- Use an inverse matrix to solve a system of linear equations.

Keywords in Section 2.3

- inverse matrix: 反矩陣
- invertible: 可逆
- nonsingular: 非奇異
- noninvertible: 不可逆
- singular: 奇異
- power: 冪次

2.4 Elementary Matrices

- Row elementary matrix:

  An $n \times n$ matrix is called an elementary matrix if it can be obtained from the identity matrix I_n by a single elementary operation.

- Three row elementary matrices:

  (1) $R_{ij} = r_{ij}(I)$ Interchange two rows.

  (2) $R_i^{(k)} = r_i^{(k)}(I)$ $(k \neq 0)$ Multiply a row by a nonzero constant.

  (3) $R_{ij}^{(k)} = r_{ij}^{(k)}(I)$ Add a multiple of a row to another row.

- Note:

  Only do <u>a single</u> elementary row operation.

- Ex 1: (Elementary matrices and nonelementary matrices)

$$(a)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Yes $(r_2^{(3)}(I_3))$

$$(b)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

No (not square)

$$(c)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

No (Row multiplication must be by a nonzero constant)

$$(d)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Yes $(r_{23}(I_3))$

$$(e)\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Yes $(r_{12}^{(2)}(I_2))$

$$(f)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

No (Use two elementary row operations)

- Thm 2.12: (Representing elementary row operations)

Let E be the elementary matrix obtained by performing an elementary row operation on I_m. If that same elementary row operation is performed on an $m \times n$ matrix A, then the resulting matrix is given by the product EA.

$$r(I) = E$$
$$r(A) = EA$$

- Notes:

(1) $r_{ij}(A) = R_{ij}A$

(2) $r_i^{(k)}(A) = R_i^{(k)}A$

(3) $r_{ij}^{(k)}(A) = R_{ij}^{(k)}A$

- Ex 2: (Elementary matrices and elementary row operation)

$$(a)\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&2&1\\1&-3&6\\3&2&-1\end{bmatrix}=\begin{bmatrix}1&-3&6\\0&2&1\\3&2&-1\end{bmatrix}(r_{12}(A)=R_{12}A)$$

$$(b)\begin{bmatrix}1&0&0\\0&\frac{1}{2}&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-4&1\\0&2&6&-4\\0&1&3&1\end{bmatrix}=\begin{bmatrix}1&0&-4&1\\0&1&3&-2\\0&1&3&1\end{bmatrix}(r_2^{(\frac{1}{2})}(A)=R_2^{(\frac{1}{2})}A)$$

$$(c)\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-1\\-2&-2&3\\0&4&5\end{bmatrix}=\begin{bmatrix}1&0&-1\\0&-2&1\\0&4&5\end{bmatrix}(r_{12}^{(2)}(A)=R_{12}^{(5)}A)$$

- Ex 3: (Using elementary matrices)

Find a sequence of elementary matrices that can be used to write the matrix A in row-echelon form.

$$A = \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

Sol:

$$E_1 = r_{12}(I_3) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2 = r_{13}^{(-2)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$E_3 = r_3^{(\frac{1}{2})}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$A_1 = r_{12}(A) = E_1 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

$$A_2 = r_{13}^{(-2)}(A_1) = E_2 A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix}$$

$$A_3 = r_3^{(\frac{1}{2})}(A_2) = E_3 A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix} = B$$

row-echelon form

$$\therefore\ B = E_3 E_2 E_1 A \quad \text{or} \quad B = r_3^{(\frac{1}{2})}(r_{13}^{(-2)}(r_{12}(A)))$$

- Row-equivalent:

Matrix B is **row-equivalent** to A if there exists a finite number of elementary matrices such that

$$B = E_k E_{k-1} \cdots E_2 E_1 A$$

- Thm 2.13: (Elementary matrices are invertible)

  If E is an elementary matrix, then E^{-1} exists and is an elementary matrix.

- Notes:

  (1) $(R_{ij})^{-1} = R_{ij}$

  (2) $(R_i^{(k)})^{-1} = R_i^{(\frac{1}{k})}$

  (3) $(R_{ij}^{(k)})^{-1} = R_{ij}^{(-k)}$

- Ex:

Elementary Matrix / Inverse Matrix

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{12} \qquad (R_{12})^{-1} = E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{12} \text{ (Elementary Matrix)}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = R_{13}^{(-2)} \qquad (R_{13}^{(-2)})^{-1} = E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = R_{13}^{(2)} \text{ (Elementary Matrix)}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = R_3^{(\frac{1}{2})} \qquad (R_3^{(\frac{1}{2})})^{-1} = E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = R_3^{(2)} \text{ (Elementary Matrix)}$$

- Thm 2.14: (A property of invertible matrices)

A square matrix A is invertible if and only if it can be written as the product of elementary matrices.

Pf: (1) Assume that A is the product of elementary matrices.

(a) Every elementary matrix is invertible.

(b) The product of invertible matrices is invertible.

Thus A is invertible.

(2) If A is invertible, $A\mathbf{x}=0$ has only the trivial solution. (Thm. 2.11)

$$\Rightarrow [A \vdots 0] \to [I \vdots 0]$$

$$\Rightarrow E_k \cdots E_3 E_2 E_1 A = I$$

$$\Rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_k^{-1}$$

Thus A can be written as the product of elementary matrices.

- Ex 4:

Find a sequence of elementary matrices whose product is

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix}$$

Sol:

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix} \xrightarrow{r_1^{(-1)}} \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \xrightarrow{r_{12}^{(-3)}} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

$$\xrightarrow{r_2^{(\frac{1}{2})}} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{r_{21}^{(-2)}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore $\quad R_{21}^{(-2)} R_2^{(\frac{1}{2})} R_{12}^{(-3)} R_1^{(-1)} A = I$

$$\text{Thus}\quad A = (R_1^{(-1)})^{-1}(R_{12}^{(-3)})^{-1}(R_2^{(\frac{1}{2})})^{-1}(R_{21}^{(-2)})^{-1}$$

$$= R_1^{(-1)}R_{12}^{(3)}R_2^{(2)}R_{21}^{(2)}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

- Note:

If A is invertible

$$\text{Then}\quad E_k \cdots E_3 E_2 E_1 A = I$$

$$A^{-1} = E_k \cdots E_3 E_2 E_1$$

$$E_k \cdots E_3 E_2 E_1 [A \vdots I] = [I \vdots A^{-1}]$$

- Thm 2.15: (Equivalent conditions)

If A is an $n \times n$ matrix, then the following statements are equivalent.

(1) A is invertible.

(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ column matrix $\mathbf{b}$.

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(4) A is row-equivalent to I_n .

(5) A can be written as the product of elementary matrices.

- *LU*-factorization:

  If the $n \times n$ matrix A can be written as the product of a lower triangular matrix L and an upper triangular matrix U, then A=LU is an LU-factorization of A

$$A = LU$$

L is a lower triangular matrix

U is an upper triangular matrix

- Note:

  If a square matrix A can be row reduced to an upper triangular matrix U using only the row operation of adding a multiple of one row to another row below it, then it is easy to find an LU-factorization of A.

$$E_k \cdots E_2 E_1 A = U$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U$$

$$A = LU$$

- Ex 5: (LU-factorization)

$$(a)\ \ A=\begin{bmatrix}1&2\\1&0\end{bmatrix}\qquad\qquad (b)\ \ A=\begin{bmatrix}1&-3&0\\0&1&3\\2&-10&2\end{bmatrix}$$

Sol: (a)

$$A=\begin{bmatrix}1&2\\1&0\end{bmatrix}\xrightarrow{r_{12}^{(-1)}}\begin{bmatrix}1&2\\0&-2\end{bmatrix}=U$$

$$\Rightarrow R_{12}^{(-1)}A=U$$

$$\Rightarrow A=(R_{12}^{(-1)})^{-1}U=LU$$

$$\Rightarrow L=(R_{12}^{(-1)})^{-1}=R_{12}^{(1)}=\begin{bmatrix}1&0\\1&1\end{bmatrix}$$

(b)

$$A=\begin{bmatrix}1&-3&0\\0&1&3\\2&-10&2\end{bmatrix}\xrightarrow{r_{13}^{(-2)}}\begin{bmatrix}1&-3&0\\0&1&3\\0&-4&2\end{bmatrix}\xrightarrow{r_{23}^{(4)}}\begin{bmatrix}1&-3&0\\0&1&3\\0&0&14\end{bmatrix}=U$$

$$\Rightarrow R_{23}^{(4)}R_{13}^{(-2)}A=U$$

$$\Rightarrow A=(R_{13}^{(-2)})^{-1}(R_{23}^{(4)})^{-1}U=LU$$

$$\Rightarrow L=(R_{13}^{(-2)})^{-1}(R_{23}^{(4)})^{-1}=R_{13}^{(2)}R_{23}^{(-4)}$$

$$=\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-4&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\2&-4&1\end{bmatrix}$$

- Solving $Ax=b$ with an LU-factorization of A:

$$Ax = b \qquad \text{If } \ A = LU\text{, then } \ LUx = b$$

$$\text{Let } \ y = Ux\text{, then } \ Ly = b$$

- Two steps:

(1) Write $y = Ux$ and solve $Ly = b$ for y

(2) Solve $Ux = y$ for x

- Ex 7: (Solving a linear system using *LU*-factorization)

$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & & & = & -5 \\ & & x_2 & + & 3x_3 & = & -1 \\ 2x_1 & - & 10x_2 & + & 2x_3 & = & -20 \end{array}$$

Sol:

$$A=\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix}=LU$$

(1) Let $y=Ux$, and solve $Ly=b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} -5 \\ -1 \\ -20 \end{bmatrix} \Rightarrow \begin{aligned} y_1 &= -5 \\ y_2 &= -1 \\ y_3 &= -20-2y_1+4y_2 \\ &= -20-2(-5)+4(-1)=-14 \end{aligned}$$

(2) Solve the following system $Ux = y$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -14 \end{bmatrix}$$

So $\quad x_3 = -1$

$$x_2 = -1 - 3x_3 = -1 - (3)(-1) = 2$$

$$x_1 = -5 + 3x_2 = -5 + 3(2) = 1$$

Thus, the solution is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Key Learning in Section 2.4

- Factor a matrix into a product of elementary matrices.
- Find and use an *LU*-factorization of a matrix to solve a system of linear equations.

Keywords in Section 2.4

- row elementary matrix: 列基本矩陣
- row equivalent: 列等價
- lower triangular matrix: 下三角矩陣
- upper triangular matrix: 上三角矩陣
- *LU*-factorization: *LU*-分解

2.5 Markov Chains

- Stochastic Matrices

$\{S_1, S_2, \ldots, S_n\}$ is a finite set of state of a given population.

$p_{ij} = 0$ is certain to not change from the jth state to the ith state.

$p_{ij} = 1$ is certain to change from the jth state to the ith state.

$0 \le p_{ij} \le 1$

$$\overbrace{S_1 \qquad S_2 \qquad \cdots \qquad S_n}^{\text{Form}}$$

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ + & + & & + \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ + & + & & + \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \left.\begin{array}{c} S_1 \\ S_2 \\ \vdots \\ S_n \end{array}\right\} \text{To}$$

$$=1 \qquad =1 \qquad\qquad =1$$

P is called the matrix of transition probabilities.

- Ex 1: (Examples of Stochastic Matrices and Nonstochastic Matrices)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
stochastic

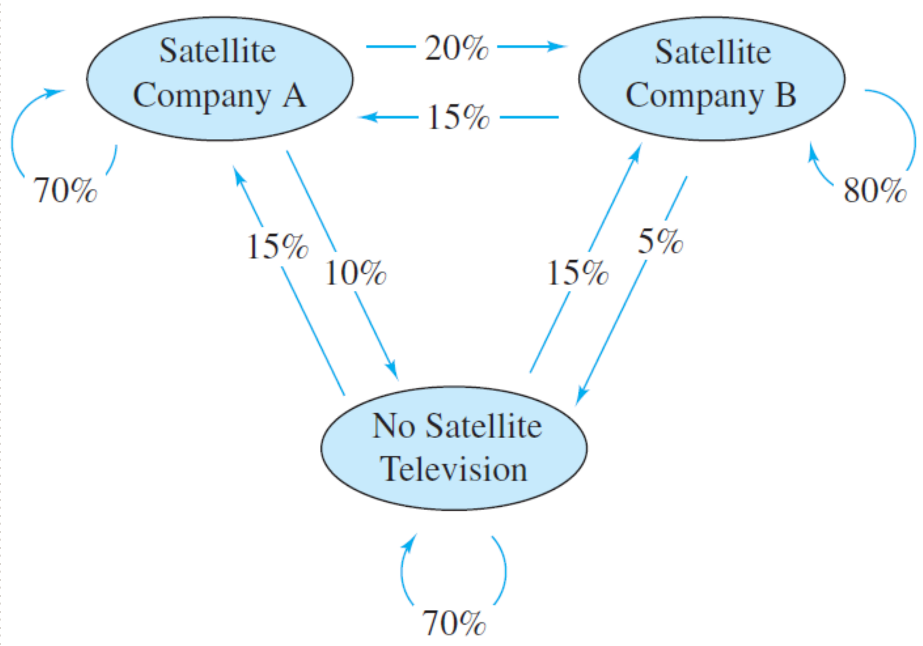
$$\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ 0.2 & 0.3 & 0.4 \\ 0.3 & 0.4 & 0.5 \end{bmatrix}$$
not stochastic

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{4} & \frac{2}{3} & 0 \end{bmatrix}$$
stochastic

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix}$$
not stochastic

- Ex 2: (A Consumer Preference Model)

$$P = \begin{array}{cccc} A & B & None & \\ \left[\begin{array}{ccc} 0.70 & 0.15 & 0.15 \\ 0.20 & 0.80 & 0.15 \\ 0.10 & 0.05 & 0.70 \end{array}\right] & & & \begin{array}{c} A \\ B \\ None \end{array} \end{array}$$



$$X = \left[\begin{array}{c} 15{,}000 \\ 20{,}000 \\ 65{,}000 \end{array}\right] \begin{array}{c} A \\ B \\ None \end{array}$$

$$PX = \left[\begin{array}{ccc} 0.70 & 0.15 & 0.15 \\ 0.20 & 0.80 & 0.15 \\ 0.10 & 0.05 & 0.70 \end{array}\right] \left[\begin{array}{c} 15{,}000 \\ 20{,}000 \\ 65{,}000 \end{array}\right] = \left[\begin{array}{c} 23{,}250 \\ 28{,}750 \\ 48{,}000 \end{array}\right] \begin{array}{c} A \\ B \\ None \end{array}$$

- Ex 3: (A Consumer Preference Model)

$$PX = \begin{bmatrix} 23{,}250 \\ 28{,}750 \\ 48{,}000 \end{bmatrix} \begin{matrix} \text{A} \\ \text{B} \\ \text{None} \end{matrix} \quad \text{After 1 year}$$

$$P^5X \approx \begin{bmatrix} 32{,}411 \\ 43{,}812 \\ 23{,}777 \end{bmatrix} \begin{matrix} \text{A} \\ \text{B} \\ \text{None} \end{matrix} \quad \text{After 5 year}$$

$$P^3X \approx \begin{bmatrix} 30{,}283 \\ 39{,}042 \\ 30{,}675 \end{bmatrix} \begin{matrix} \text{A} \\ \text{B} \\ \text{None} \end{matrix} \quad \text{After 3 year}$$

$$P^{10}X \approx \begin{bmatrix} 33{,}287 \\ 47{,}147 \\ 19{,}566 \end{bmatrix} \begin{matrix} \text{A} \\ \text{B} \\ \text{None} \end{matrix} \quad \text{After 10 year}$$

$$\overline{X} \approx \begin{bmatrix} 33{,}333 \\ 47{,}619 \\ 19{,}048 \end{bmatrix} \begin{matrix} \text{A} \\ \text{B} \\ \text{None} \end{matrix} \quad \text{Steady state matrix}$$

$$P\overline{X} = \begin{bmatrix} 0.70 & 0.15 & 0.15 \\ 0.20 & 0.80 & 0.15 \\ 0.10 & 0.05 & 0.70 \end{bmatrix} \begin{bmatrix} 33{,}333 \\ 47{,}619 \\ 19{,}048 \end{bmatrix} \approx \begin{bmatrix} 33{,}333 \\ 47{,}619 \\ 19{,}048 \end{bmatrix} = \overline{X}$$

EXAMPLE 1 Stochastic matrices are used by city planners to analyze trends in land use. Such a matrix has been used by the city of Toronto, for example. The researchers collect data and write them in the form of a stochastic matrix P. The rows and columns of P represent land uses. We illustrate typical categories for a five-year period in the matrix that follows. The element p_{ij} is the probability that land that was in use j in 2000 was in use i in 2005.

$$\begin{array}{c} \text{Use in 2000} \\ \begin{array}{ccccc} 1 & \;\;2 & \;\;3 & \;\;4 & \;\;5 \end{array} \\ \left[\begin{array}{ccccc} .4 & .15 & .1 & .05 & .05 \\ .1 & .35 & .3 & .15 & .35 \\ .15 & .15 & .5 & .35 & .20 \\ .1 & \boxed{.30} & .1 & .4 & .25 \\ .25 & .05 & 0 & .05 & .15 \end{array}\right] \end{array} \quad \begin{array}{l} \text{Use in 2005} \\ \text{1. Residential} \\ \text{2. Office} \\ \text{3. Commercial} \\ \text{4. Parking} \\ \text{5. Vacant} \end{array}$$

Let us interpret some of the information contained in this matrix. For example, $p_{42} = 0.30$. This tells us that land that was office space in 2000 had a probability of 0.30 of becoming parking area by 2005. The fourth row of P gives the probabilities that various areas of the city have become parking areas by 2005. These relatively large figures reveal the increasingly dominant role of parking in land use.

Example 2: Population Movement

- In 2007, 82 million of people live in cities and 163 million of people live in the surrounding suburbs. Represent this information by the matrix

$$X_0 = \begin{bmatrix} 82 \\ 163 \end{bmatrix}$$

- The probability of a person who stayed in the city in 2007, will be staying in the city in the next year (2008) is 0.96. Thus the probability of moving to the suburbs is 0.04.
- The probability of a person who stayed in the suburb in 2007, will be moving to the city next year is 0.01; the probability of remaining in suburb is then 0.99.

$$\begin{array}{ccc} & \text{(from)} & \text{(to)} \\ & \begin{array}{cc} \text{city} & \text{suburb} \end{array} & \\ P = & \begin{bmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{bmatrix} & \begin{array}{c} \text{city} \\ \text{suburb} \end{array} \end{array}$$

Example 2: (cont'd)

$$X_0 = \begin{bmatrix} 82 \\ 163 \end{bmatrix}$$

- City population in 2008 (1 year after)
  = people who remained from 2007 + people who moved in from the suburbs
  $= (0.96 \times 82) + (0.01 \times 163) = 80.35$ million
- Suburban population in 2008 (1 year after)
  = people who moved in from the city + people who stayed from 2007
  $= (0.04 \times 82) + (0.99 \times 163) = 164.65$ million
- Can arrive at these numbers using matrix multiplication

$$\boxed{X_1 = PX_0} = \begin{bmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{bmatrix} \begin{bmatrix} 82 \\ 163 \end{bmatrix} = \begin{bmatrix} 80.35 \\ 164.65 \end{bmatrix}$$

Using 2007 as the base year, let X_1 be the population in 2008, one year later. We can write

$$\boxed{X_1 = PX_0}$$

Assume that the population flow represented by the matrix P is unchanged over the years. The population distribution X_2 after 2 years is given by

$$\boxed{X_2 = PX_1}$$

After 3 years the population distribution is given by

$$\boxed{X_3 = PX_2}$$

After n years we get

$$\boxed{X_n = PX_{n-1}}$$

The predictions of this model (to four decimal places) are

$$X_0 = \begin{array}{cl} \begin{bmatrix} 82 \\ 163 \end{bmatrix} & \begin{array}{l} \text{city} \\ \text{suburb} \end{array} \end{array}, \quad X_1 = \begin{bmatrix} 80.35 \\ 164.65 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 78.7825 \\ 166.2175 \end{bmatrix},$$

$$X_3 = \begin{bmatrix} 77.2934 \\ 167.7066 \end{bmatrix}, \quad X_4 = \begin{bmatrix} 75.8787 \\ 169.1213 \end{bmatrix}, \longrightarrow$$ *Will there be a steady-state result?*

and so on.

Observe how the city population is decreasing annually, while that of the suburbs is increasing. We return to this model in section 3.5. There we find that the sequence $X_0, X_1, X_2, \ldots$ approaches $\begin{bmatrix} 49 \\ 196 \end{bmatrix}$. If conditions do not change, city population will gradually approach 49 million, while the population of suburbia will approach 196 million.

Further, note that the sequence $X_1, X_2, X_3, \ldots X_n$ can be directly computed from X_0, as follows.

$$X_1 = PX_0, \qquad X_2 = P^2X_0, \qquad X_3 = P^3X_0, \ldots, X_n = P^nX_0$$

The matrix P^n is a stochastic matrix that takes X_0 into X_n, in n steps. This result can be generalized. That is, P^n can be used in this manner to predict the distribution n stages later, from any given distribution.

$$X_{i+n} = P^n X_i$$

An n-step transition matrix

- Steady state matrix:

$$P\bar{X} = \bar{X}$$

The matrix X_n eventually reaches a steady state. That is, as long as the matrix P does not change, the matrix product P^nX approaches a limit $\bar{X}$. The limit is the ***steady state matrix***.

- Regular stochastic matrix:

A stochastic matrix P is regular when some power of P has only positive entries.

- Note:

When P is a regular stochastic matrix, the corresponding **regular Markov chain**

$$PX_0, P^2X_0, P^3X_0, \ldots$$

approaches a unique **steady state matrix** $\bar{X}$.

- Ex 4: (Regular Stochastic Matrices)

(a) The stochastic matrix

$$P=\begin{bmatrix}0.70 & 0.15 & 0.15\\ 0.20 & 0.80 & 0.15\\ 0.10 & 0.05 & 0.70\end{bmatrix}$$

is regular because P has only positive entries.

(b) The stochastic matrix

$$P=\begin{bmatrix}0.50 & 1.00\\ 0.50 & 0\end{bmatrix}$$

has only positive entries.

- Ex 4: (Regular Stochastic Matrices)

(c) The stochastic matrix

$$P = \begin{bmatrix} \frac{1}{3} & 0 & 1 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 0 \end{bmatrix}$$

is not regular because every power of P has two zeros in its second column.

- Ex 5: (Finding a Steady State Matrix)

Find the steady state matrix X of the Markov chain whose matrix of transition probabilities is the regular matrix

$$P=\begin{bmatrix}0.70 & 0.15 & 0.15\\ 0.20 & 0.80 & 0.15\\ 0.10 & 0.05 & 0.70\end{bmatrix}$$

Sol:

Letting $\overline{X}=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$. Then use the matrix equation $P\overline{X}=\overline{X}$ to obtain

$$\begin{bmatrix}0.70 & 0.15 & 0.15\\ 0.20 & 0.80 & 0.15\\ 0.10 & 0.05 & 0.70\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

or

$$\begin{aligned}
0.70x_1 + 0.15x_2 + 0.15x_3 &= x_1 \\
0.20x_1 + 0.80x_2 + 0.15x_3 &= x_2 \\
0.10x_1 + 0.05x_2 + 0.70x_3 &= x_3
\end{aligned}$$

Use these equations and the fact that $x_1 + x_2 + x_3 = 1$ to write the system of linear equations below.

$$\begin{aligned}
-0.30x_1 + 0.15x_2 + 0.15x_3 &= 0 \\
0.20x_1 - 0.20x_2 + 0.15x_3 &= 0 \\
0.10x_1 + 0.05x_2 - 0.30x_3 &= 0 \\
x_1 + \qquad x_2 + \qquad x_3 &= 1
\end{aligned}$$

- Ex 5: (Finding a Steady State Matrix)

Use any appropriate method to verify that the solution of this system is

$$x_1 = \frac{1}{3}, \quad x_2 = \frac{10}{21} \quad \text{and} \quad x_3 = \frac{4}{21}$$

So the steady state matrix is

$$\overline{X} = \begin{bmatrix} \frac{1}{3} \\ \frac{10}{21} \\ \frac{4}{21} \end{bmatrix} \approx \begin{bmatrix} 0.3333 \\ 0.4762 \\ 0.1905 \end{bmatrix}$$

Check: $P\overline{X} = \overline{X}$

- Finding the Steady State Matrix of a Markov chain:

1. Check to see that the matrix of transition probabilities P is a regular matrix.

2. Solve the system of linear equations obtained from the matrix equation $P\overline{X} = \overline{X}$ along with the equation $x_1 + x_2 + \ldots + x_n = 1$

3. Check the solution found in Step 2 in the matrix equation $P\overline{X} = \overline{X}$

- Absorbing state:

Consider a Markov chain with n different states $\{S_1, S_2, \ldots, S_n\}$. The ith state S_i is an **absorbing state** when, in the matrix of transition probabilities P, $p_{ii} = 1$. That is, the entry on the main diagonal of P is 1 and all other entries in the ith column of P are 0.

- Absorbing Markov chain:

An **absorbing Markov chain** has the two properties listed below.

1. The Markov chain has at least one absorbing state.

2. It is possible for a member of the population to move from any nonabsorbing state to an absorbing state in a finite number of transitions.

- Ex 6: (Absorbing and Nonabsorbing Markov Chains)

(a) For the matrix

$$\begin{array}{cc} & \overbrace{\begin{array}{ccc} S_1 & S_2 & S_3 \end{array}}^{\text{From}} & \\ P = & \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 1 & 0.5 \\ 0.6 & 0 & 0.5 \end{bmatrix} & \left.\begin{array}{c} S_1 \\ S_2 \\ S_3 \end{array}\right\} \text{To} \end{array}$$

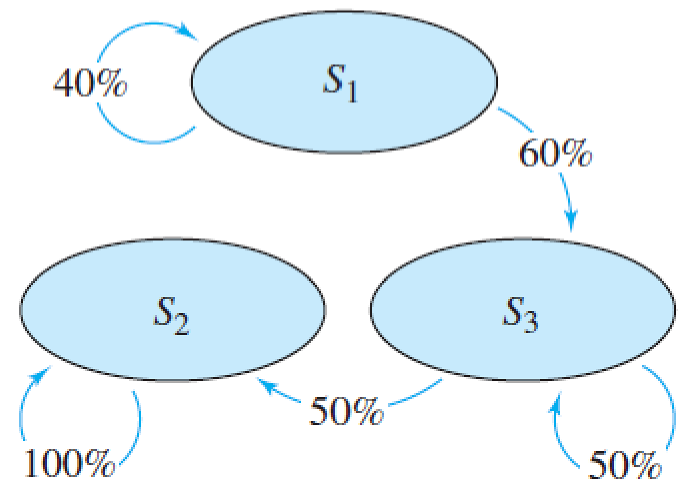


Figure 2.2

the second state, represented by the second column, is absorbing. Moreover, the corresponding Markov chain is also absorbing because it is possible to move from S_1 to S_2 in two transitions, and it is possible to move from S_3 to S_2 in one transition.

(b) For the matrix

$$P = \begin{array}{c} \\ \\ \end{array}\begin{array}{cl} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} & \text{From} \\ \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0.5 & 1 & 0.5 & 0 \\ 0 & 0 & 0.4 & 0.5 \\ 0 & 0 & 0.6 & 0.5 \end{bmatrix} & \left.\begin{array}{c} S_1 \\ S_2 \\ S_3 \\ S_4 \end{array}\right\} \text{To} \end{array}$$

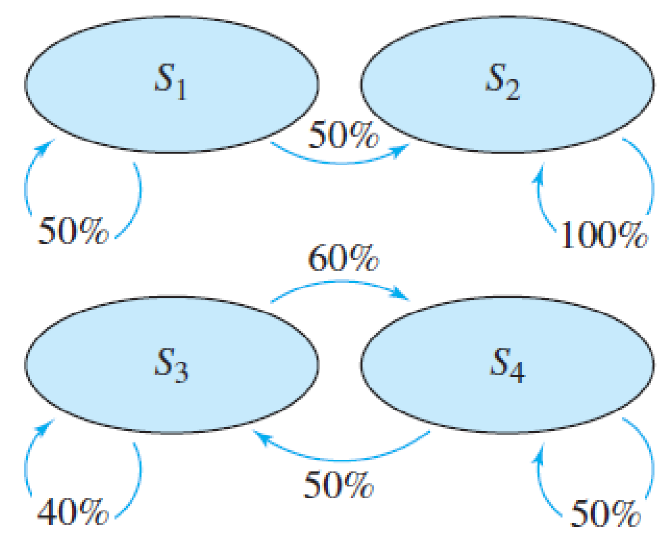


Figure 2.3

the second state is absorbing. However, the corresponding Markov chain is not absorbing because there is no way to move from state S_3 or state S_4 to state S_2.

- Ex 6: (Absorbing and Nonabsorbing Markov Chains)

(a) For the matrix

$$P=\begin{array}{c} \overbrace{\begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array}}^{\text{From}} \\ \begin{bmatrix} 0.5 & 0 & 0.2 & 0 \\ 0.2 & 1 & 0.3 & 0 \\ 0.1 & 0 & 0.4 & 0 \\ 0.2 & 0 & 0.1 & 0 \end{bmatrix} \end{array} \left.\begin{array}{c} S_1 \\ S_2 \\ S_3 \\ S_4 \end{array}\right\} \text{To}$$

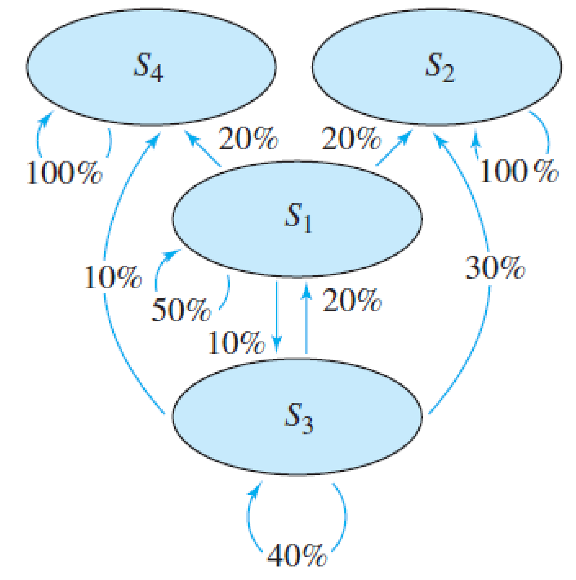


Figure 2.4

has two absorbing states: S_2 and S_4. Moreover, the corresponding Markov chain is also absorbing because it is possible to move from either of the nonabsorbing states, S_1 or S_3, to either of the absorbing states in one step.

Ex 7: (Finding Steady State Matrices of Absorbing Markov Chains)

$$\text{(a) } P = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 1 & 0.5 \\ 0.6 & 0 & 0.5 \end{bmatrix}$$

Use the matrix equation $P\bar{X} = \bar{X}$, or

$$\begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 1 & 0.5 \\ 0.6 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

along with the equation $x_1 + x_2 + x_3 = 1$ to write the system of linear equations

$$\begin{aligned}
-0.6x_1 \qquad\qquad\qquad\qquad &= 0 \\
0.5x_3 &= 0 \\
0.6x_1 \qquad\quad - 0.5x_3 &= 0 \\
x_1 + \quad x_2 + \quad x_3 &= 1
\end{aligned}$$

The solution of this system is $x_1 = 0$, $x_2 = 1$, and $x_3 = 0$, so the steady state matrix is $X = [0\ 1\ 0]^T$. Note that $\overline{X}$ coincides with the second column of the matrix of transition probabilities P.

Ex 7: (Finding Steady State Matrices of Absorbing Markov Chains)

$$\text{(b) } P = \begin{bmatrix} 0.5 & 0 & 0.2 & 0 \\ 0.2 & 1 & 0.3 & 0 \\ 0.1 & 0 & 0.4 & 0 \\ 0.2 & 0 & 0.1 & 1 \end{bmatrix}$$

Use the matrix equation $P\overline{X} = \overline{X}$, or

$$\begin{bmatrix} 0.5 & 0 & 0.2 & 0 \\ 0.2 & 1 & 0.3 & 0 \\ 0.1 & 0 & 0.4 & 0 \\ 0.2 & 0 & 0.1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

along with the equation $x_1 + x_2 + x_3 + x_4 = 1$ to write the system of linear equations

$$
\begin{aligned}
-0.6x_1 \qquad\quad + 0.2x_3 \qquad\quad &= 0 \\
0.2x_1 \qquad\quad + 0.3x_3 \qquad\quad &= 0 \\
0.6x_1 \qquad\quad - 0.6x_3 \qquad\quad &= 0 \\
0.2x_1 \qquad\quad + 0.1x_3 \qquad\quad &= 0 \\
x_1 + x_2 + \qquad x_3 + x_4 &= 1
\end{aligned}
$$

The solution of this system is $x_1 = 0$, $x_2 = 1 - t$, $x_3 = 0$, and $x_4 = t$, where t is any real number such that $0 \le x \le 1$. So, the steady matrix is $\overline{X} = [0 \;\; 1 - t \;\; 0 \;\; t]^T$. The Markov chain has an infinite number of steady state matrices.

Key Learning in Section 2.5

- Use a stochastic matrix to find the *n*th state matrix of a Markov chain.
- Find the steady state matrix of a Markov chain.
- Find the steady state matrix of an absorbing Markov chain.

Keywords in Section 2.5

- matrix of transition probabilities: 轉移機率矩陣
- stochastic: 隨機
- stochastic matrix: 隨機矩陣
- state matrix: 狀態矩陣
- Markov chain: 馬可夫鏈
- steady state: 穩定狀態
- regular stochastic matrix: 正規隨機矩陣
- regular Markov chain: 正規馬可夫鏈
- steady state matrix: 穩定狀態矩陣
- absorbing Markov chains: 吸收馬可夫鏈

2.6 More Applications of Matrix Operations

- Cryptography

a method of using matrix multiplication to **encode** and **decode** messages.

0 = __	7 = G	14 = N	21 = U
1 = A	8 = H	15 = O	22 = V
2 = B	9 = I	16 = P	23 = W
3 = C	10 = J	17 = Q	24 = X
4 = D	11 = K	18 = R	25 = Y
5 = E	12 = L	19 = S	26 = Z
6 = F	13 = M	20 = T	

- Ex 1: (Forming Uncoded Row Matrices)

$$\begin{matrix}[13 & 5 & 5] & [20 & 0 & 13] & [5 & 0 & 13] & [15 & 14 & 4] & [1 & 25 & 0] \\ M & E & E & T & \_ & M & E & \_ & M & O & N & D & A & Y & \_\end{matrix}$$

- Notes:

(1) The use of a blank space fill out the last uncoded row matrix.

(2) To encode a message, choose an $n \times n$ invertible matrix A and multiply the uncoded row matrices (on the right) by A to obtain coded row matrices.

- Ex 2: (Encoding a Message)

$$A=\begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix}$$

Uncoded Row Matrix	Encoding Matrix *A*	Coded Row Matrix

$$\begin{bmatrix}13 & 5 & 5\end{bmatrix}\begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix}=\begin{bmatrix}13 & -26 & 21\end{bmatrix}$$

$$\begin{bmatrix}20 & 0 & 13\end{bmatrix}\begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix}=\begin{bmatrix}33 & -53 & -12\end{bmatrix}$$

Uncoded Row Matrix	Encoding Matrix A	Coded Row Matrix

$$\begin{bmatrix} 5 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 18 & -23 & -42 \end{bmatrix}$$

$$\begin{bmatrix} 15 & 14 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 5 & -20 & 56 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 25 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} -24 & 23 & 77 \end{bmatrix}$$

the sequence of coded row matrices

$$[13 \quad -26 \quad 21][33 \quad -53 \quad -12][18 \quad -23 \quad -42][5 \quad -20 \quad 56][-24 \quad 23 \quad 17]$$

cryptogram

$$13 \ -26 \ 21 \ 33 \ -53 \ -12 \ 18 \ -23 \ -42 \ 5 \ -20 \ -56 \ -24 \ 23 \ 17$$

an uncoded $1 \times n$ matrix

$$X = [x_1 \quad x_2 \quad \cdots \quad x_n]$$

$Y = XA$ is the corresponding encoded matrix

to obtain

$$YA^{-1} = (XA)A^{-1} = X$$

- Ex 3: (Decoding a Message)

$$A=\begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix}$$

$$13\ -26\ 21\ 33\ -53\ -12\ 18\ -23\ -42\ 5\ -20\ -56\ -24\ 23\ 17$$

$$\left[\begin{array}{ccc|ccc}1 & -2 & 2 & 1 & 0 & 0\\ -1 & 1 & 3 & 0 & 1 & 0\\ 1 & -1 & -4 & 0 & 0 & 1\end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc}1 & 0 & 0 & -1 & -10 & -8\\ 0 & 1 & 0 & -1 & -6 & -5\\ 0 & 0 & 1 & 0 & -1 & -1\end{array}\right]$$

Gauss-Jordan eliminiation

the sequence of coded row matrices

$$[13\quad -26\quad 21][33\quad -53\quad -12][18\quad -23\quad -42][5\quad -20\quad 56][-24\quad 23\quad 17]$$

Coded Row Matrix | Encoding Matrix A^{-1} | Decoded Row Matrix

$$\begin{bmatrix}13 & -26 & 21\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}13 & 5 & 5\end{bmatrix}$$

$$\begin{bmatrix}33 & -53 & -12\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}20 & 0 & 13\end{bmatrix}$$

$$\begin{bmatrix}18 & -23 & -42\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}5 & 0 & 13\end{bmatrix}$$

$$\begin{bmatrix}5 & -20 & 56\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}15 & 14 & 4\end{bmatrix}$$

$$\begin{bmatrix}-24 & 23 & 77\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}1 & 25 & 0\end{bmatrix}$$

Coded Row Matrix | Encoding Matrix A^{-1} | Decoded Row Matrix

$$\begin{bmatrix} 13 & -26 & 21 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 13 & 5 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 33 & -53 & -12 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 20 & 0 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 18 & -23 & -42 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -20 & 56 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 15 & 14 & 4 \end{bmatrix}$$

Coded Row Matrix	Encoding Matrix A^{-1}	Decoded Row Matrix

$$\begin{bmatrix} -24 & 23 & 77 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 25 & 0 \end{bmatrix}$$

the sequence of decoded row matrices

$$\begin{bmatrix} 13 & 5 & 5 \end{bmatrix}\begin{bmatrix} 20 & 0 & 13 \end{bmatrix}\begin{bmatrix} 5 & 0 & 13 \end{bmatrix}\begin{bmatrix} 15 & 14 & 4 \end{bmatrix}\begin{bmatrix} 1 & 25 & 0 \end{bmatrix}$$

the message

13	5	5	20	0	13	5	0	13	15	14	4	1	25	0
M	*E*	*E*	*T*	_	*M*	*E*	_	*M*	*O*	*N*	*D*	*A*	*Y*	_

- Input-output matrix:

$$D = \begin{array}{c} \overbrace{\begin{array}{cccc} I_1 & I_2 & \cdots & I_n \end{array}}^{\text{User (Output)}} \\ \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{bmatrix} \end{array} \quad \left.\begin{array}{c} I_1 \\ I_2 \\ \vdots \\ I_n \end{array}\right\} \text{Supplier (Input)}$$

(1) d_{ij} be the amount of output the jth industry needs from the ith industry to produce one unit of output per year.

(2) The values of d_{ij} must satisfy $0 \le d_{ij} \le 1$ and the sum of the entries in any column must be less than or equal to 1.

- Ex 4: (Forming an Input-Output Matrix)

Consider a simple economic system consisting of three industries: electricity, water, and coal. Production, or output, of one unit of electricity requires 0.5 unit of itself, 0.25 unit of water, and 0.25 unit of coal. Production of one unit of water requires 0.1 unit of electricity, 0.6 unit of itself, and 0 units of coal. Production of one unit of coal requires 0.2 unit of electricity, 0.15 unit of water, and 0.5 unit of itself. Find the input-output matrix for this system.

Sol:

The column entries show the amounts each industry requires from the others, and from itself, to produce one unit of output.

- Ex 4: (Forming an Input-Output Matrix)

$$P = \begin{array}{c} \\ \\ \\ \\ \end{array}\overbrace{\begin{array}{ccc} \text{E} & \text{W} & \text{C} \end{array}}^{\text{User (Output)}}$$

$$P = \begin{bmatrix} 0.5 & 0.1 & 0.2 \\ 0.25 & 0.6 & 0.15 \\ 0.25 & 0 & 0.5 \end{bmatrix} \left.\begin{array}{l} \text{E} \\ \text{W} \\ \text{C} \end{array}\right\} \text{Supplier (Input)}$$

The row entries show the amounts each industry supplies to the others, and to itself, for that industry to produce one unit of output. For instance, the electricity industry supplies 0.5 unit to itself, 0.1 unit to water, and 0.2 unit to coal.

- Leontief input-output model:

$$\begin{array}{cc} & \overbrace{\begin{array}{cccc} I_1 & I_2 & \cdots & I_n \end{array}}^{\text{User (Output)}} \\ D = & \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{bmatrix} \end{array} \begin{array}{c} I_1 \\ I_2 \\ \vdots \\ I_n \end{array} \left.\vphantom{\begin{array}{c} I_1 \\ I_2 \\ \vdots \\ I_n \end{array}}\right\} \text{Supplier (Input)}$$

- Closed system:

  Let the total output of the ith industry be denoted by x_i. If the economic system is **closed** (that is, the economic system sells its products only to industries within the system, as in the example above), then the total output of the ith industry is

$$x_i = d_{i1}x_1 + d_{i2}x_2 + \dots + d_{in}x_n \qquad \text{(Closed System)}$$

- Open system:

If the industries within the system sell products to nonproducing groups (such as governments or charitable organizations) outside the system, then the system is **open** and the total output of the *i*th industry is

$$x_i = d_{i1}x_1 + d_{i2}x_2 + \dots + d_{in}x_n + e_i \quad \text{(Open system)}$$

where e_i represents the external demand for the *i*th industry's product. The system of n linear equations below represents the collection of total outputs for an open system.

$$\begin{aligned} x_1 &= d_{11}x_1 + d_{12}x_2 + \dots + d_{1n}x_n + e_1 \\ x_2 &= d_{21}x_1 + d_{22}x_2 + \dots + d_{2n}x_n + e_2 \\ &\dots \\ x_n &= d_{n1}x_1 + d_{n2}x_2 + \dots + d_{nn}x_n + e_n \end{aligned}$$

The matrix form of this system is $X = DX + E$, where X is the **output matrix** and E is the **external demand matrix.**

Ex 5: (Solving for the output Matrix of an open system)

An economic system composed of three industries has the input-output matrix shown below.

$$\begin{array}{c} \\ \\ P = \end{array} \begin{array}{c} \overbrace{\begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array}}^{\text{User (Output)}} \\ \begin{bmatrix} 0.1 & 0.43 & 0 \\ 0.15 & 0 & 0.37 \\ 0.23 & 0.03 & 0.02 \end{bmatrix} \end{array} \begin{array}{c} \\ \left.\begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array}\right\} \text{Supplier (Input)} \end{array}$$

Sol:

Letting I be the identity matrix, write the equation $X = DX + E$ as $IX - DX = E$, which means that $(I - D)X = E$. Using the matrix D above produces

Ex 5: (Solving for the output Matrix of an open system)

$$I-D=\begin{bmatrix} 0.9 & -0.43 & 0 \\ -0.15 & 1 & -0.37 \\ -0.23 & -0.03 & 0.98 \end{bmatrix}$$

Using Gauss-Jordan elimination,

$$(I-D)^{-1}\approx\begin{bmatrix} 1.25 & 0.55 & 0.21 \\ 0.30 & 1.14 & 0.43 \\ 0.30 & 0.16 & 1.08 \end{bmatrix}$$

So, the output matrix is

$$X=(I-D)^{-1}E\approx\begin{bmatrix} 1.25 & 0.55 & 0.21 \\ 0.30 & 1.14 & 0.43 \\ 0.30 & 0.16 & 1.08 \end{bmatrix}\begin{bmatrix} 20{,}000 \\ 30{,}000 \\ 25{,}000 \end{bmatrix}=\begin{array}{c} \begin{bmatrix} 46{,}750 \\ 50{,}950 \\ 37{,}800 \end{bmatrix} \end{array}\begin{array}{c} \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \end{array}$$

Ex 5: (Solving for the output Matrix of an open system)

To produce the given external demands, the outputs of the three industries must be approximately 46,750 units for industry A, 50,950 units for industry B, and 37,800 units for industry C.

- Least Squares Regression analysis

  A procedure used in statistics to develop linear models.

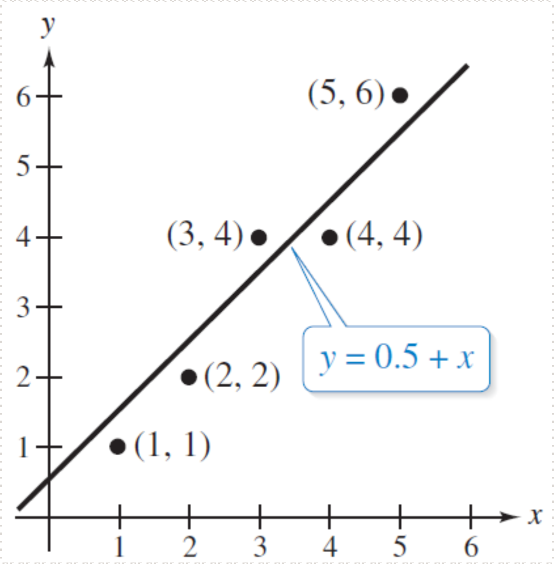
  A method for approximating a line of best fit for a given set of data points.

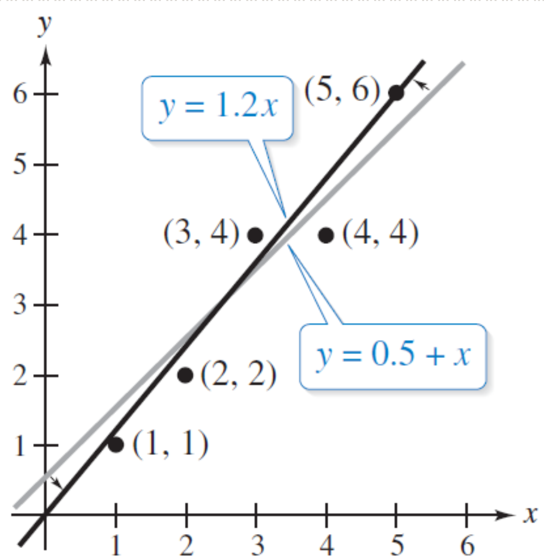
- Ex 6: (A Visual Straight-Line Approximation)

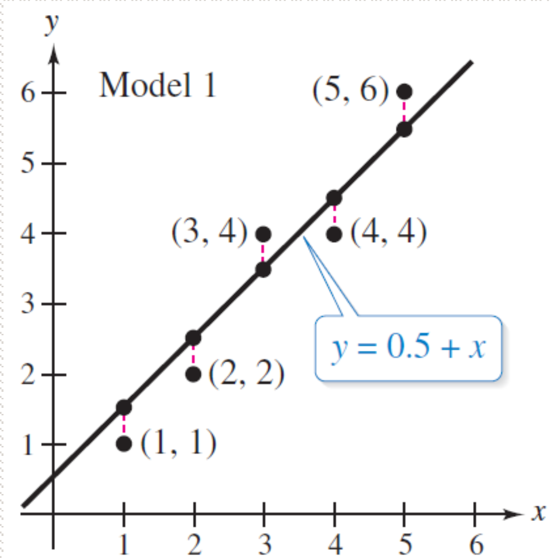
  Determine a line that appears to best fit the points

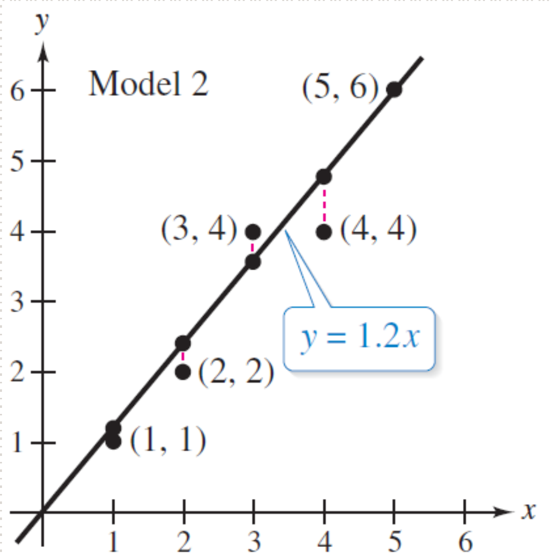
  (1, 1), (2, 2), (3, 4), (4, 4), and (5, 6).

$$y = 0.5 + x$$









Model 1: f(x) = 0.5 + x				***Model 2: f(x) = 1.2x***			
x_i	y_i	$F(x_i)$	$[y_i - F(x_i)]^2$	x_i	y_i	$F(x_i)$	$[y_i - F(x_i)]^2$
1	1	1.5	$(-0.5)^2$	1	1	1.2	$(-0.2)^2$
2	2	2.5	$(-0.5)^2$	2	2	2.4	$(-0.4)^2$
3	4	3.5	$(+0.5)^2$	3	4	3.6	$(+0.5)^2$
4	4	4.5	$(-0.5)^2$	4	4	4.8	$(-0.8)^2$
5	6	5.5	$(+0.5)^2$	5	6	6.0	$(0.0)^2$
Sum			1.25	Sum			1.00

sum of squared error

- Notes:

(1) The sums of squared errors confirm that the second model fits the given points better than the first model.

(2) Of all possible linear models for a given set of points, the model that has the best fit is defined to be the one that minimizes the sum of squared error.

(3) This model is called the least squares regression line, and the procedure for finding it is called the method of least squares.

Definition of Least Squares Regression Line

a set of points

$$(x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)$$

the least squares regression line

$$f(x) = a_0 + a_1 x$$

minimizes the sum of squared error

$$[y_1 - f(x_1)]^2 + [y_2 - f(x_2)]^2 + \cdots [y_n - f(x_n)]^2$$

the system of linear equations

$$\begin{aligned} y_1 &= f(x_1) + [y_1 - f(x_1)] \\ y_2 &= f(x_2) + [y_2 - f(x_2)] \\ &\vdots \\ y_n &= f(x_n) + [y_n - f(x_n)] \end{aligned}$$

the error

$$e_i = y_i - f(x_i)$$

the system of linear equations

$$\begin{aligned} y_1 &= (a_0 + a_1 x_1) + e_1 \\ y_2 &= (a_0 + a_1 x_2) + e_2 \\ &\vdots \\ y_n &= (a_0 + a_1 x_n) + e_n \end{aligned}$$

define *Y*, *X*, *A*, and *E*

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \qquad A = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \qquad E = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

the matrix equations

$$Y = XA + E$$

- Notes:

  (1) The matrix has a column of 1's (corresponding to a_0) and a column containing the x_i's.

  (2) This matrix equation can be used to determine the coefficients of the least squares regression line.

- Matrix From for Linear Regression

  the regression model

  $$Y = XA + E$$

  the least squares regression line

  $$A = (X^T X)^{-1} X^T Y$$

  the sum of squared error

  $$E^T E$$

- Ex 7: (Finding the Least Squares Regression Line)

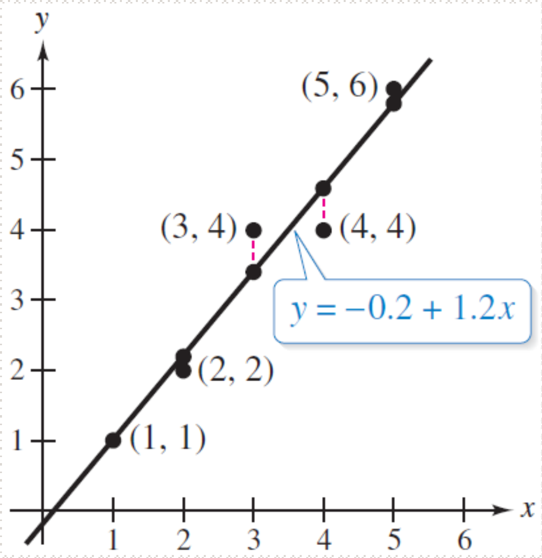
Find the least squares regression line for the points

(1, 1), (2, 2), (3, 4), (4, 4), and (5, 6).

Sol: Choose a fourth-degree polynomial function

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} \qquad Y = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \\ 6 \end{bmatrix}$$

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix}$$



$$X^T Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 17 \\ 63 \end{bmatrix}$$

the coefficient matrix

$$A = (X^T X)^{-1} X^T Y = \frac{1}{15} \begin{bmatrix} 55 & -15 \\ -15 & 5 \end{bmatrix} \begin{bmatrix} 17 \\ 63 \end{bmatrix} = \begin{bmatrix} -0.2 \\ 1.2 \end{bmatrix}$$

the least squares regression line

$$y = -0.2 + 1.2x$$

Key Learning in Section 2.6

- Use matrix multiplication to encode and decode messages.
- Use matrix algebra to analyze an economic system (Leontief input-output model).
- Find the least squares regression line for a set of data.

Keywords in Section 2.6

- cryptogram: 密碼學
- encode: 編碼
- decode: 解碼
- uncoded row matrices: 未編碼的列矩陣
- coded row matrices: 已編碼的列矩陣
- input: 輸入
- output: 輸出
- input-output matrix: 輸入-輸出矩陣
- closed: 封閉的
- open: 開放的
- external demand matrix: 外部需求矩陣
- sum of squared error: 誤差平方
- least squares regression line: 最小平方回歸線
- method of least squares: 最小平方法

- Fight Crew Scheduling

Many real-life applications of linear systems involve enormous numbers of equations and variables. For example, a flight crew scheduling problem for American Airlines required the manipulation of matrices with 837 rows and more than 12,750,000 columns. To solve this application of linear programming, researchers partitioned the problem into smaller pieces and solved it on a computer.

Information Retrieval

Information retrieval systems such as Internet search engines make use of matrix theory and linear algebra to keep track of, for instance, keywords that occur in a database. To illustrate with a simplified example, suppose you wanted to perform a search on some of the m available keywords in a database of n documents. You could represent the occurrences of the m keywords in the n documents with A, an $m \times n$ matrix in which an entry is 1 if the keyword occurs in the document and 0 if it does not occur in the document. You could represent the search with the $m \times 1$ column matrix $\mathbf{x}$ in which a 1 entry represents a keyword you are searching and 0 represents a keyword you are not searching. Then, the $n \times 1$ matrix product $A^T\mathbf{x}$ would represent the number of keywords in your search that occur in each of the n documents. For a discussion on the PageRank algorithm that is used in Google's search engine, see Section 2.5 (page 86).

- ## Beam Deflection

Recall Hooke's law, which states that for relatively small deformations of an elastic object, the amount of deflection is directly proportional to the force causing the deformation. In a simply supported elastic beam subjected to multiple forces, deflection **d** is related to force **w** by the matrix equation

$$\mathbf{d} = F\mathbf{w}$$

where is a flexibility matrix whose entries depend on the material of the beam. The inverse of the flexibility matrix, F^{-1} is called the *stiffness matrix*. In Exercises 61 and 62, you are asked to find the stiffness matrix F^{-1} and the force matrix **w** for a given set of flexibility and deflection matrices.

- Computational Fluid Dynamics

Computational fluid dynamics (CFD) is the computer-based analysis of such real-life phenomena as fluid flow, heat transfer, and chemical reactions. Solving the conservation of energy, mass, and momentum equations involved in a CFD analysis can involve large systems of linear equations. So, for efficiency in computing, CFD analyses often use matrix partitioning and *LU*-factorization in their algorithms. Aerospace companies such as Boeing and Airbus have used CFD analysis in aircraft design. For instance, engineers at Boeing used CFD analysis to simulate airflow around a virtual model of their 787 aircraft to help produce a faster and more efficient design than those of earlier Boeing aircraft.

Google's PageRank algorithm makes use of Markov chains. For a search set that contains n web pages, define an $n \times n$ matrix A such that $a_{ij} = 1$ when page j references page i and $a_{ij} = 0$ otherwise. Adjust A to account for web pages without external references, scale each column of A so that A is stochastic, and call this matrix B. Then define

$$M = pB + \frac{1-p}{n}E$$

where p is the probability that a user follows a link on a page, $1 - p$ is the probability that the user goes to any page at random, and E is an $n \times n$ matrix whose entries are all 1. The Markov chain whose matrix of transition probabilities is M converges to a unique *steady state matrix*, which gives an estimate of page ranks. Section 10.3 discusses a method that can be used to estimate the steady state matrix.

- Data Encryption

Information security is of the utmost importance when conducting business online. If a malicious party should receive confidential information such as passwords, personal identification numbers, credit card numbers, Social Security numbers, bank account details, or sensitive company information, then the effects can be damaging. To protect the confidentiality and integrity of such information, Internet security can include the use of data *encryption*, the process of encoding information so that the only way to decode it, apart from an "exhaustion attack," is to use a *key*. Data encryption technology uses algorithms based on the material presented here, but on a much more sophisticated level, to prevent malicious parties from discovering the key.