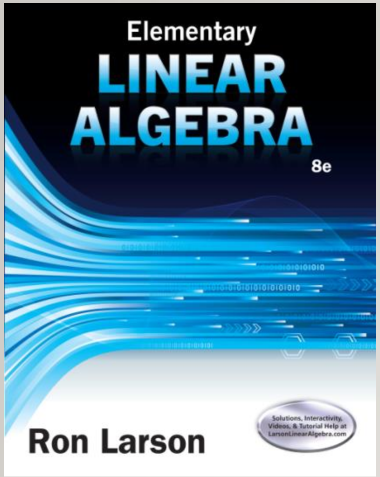
CENGAGE Learning



CHAPTER 3
DETERMINANTS

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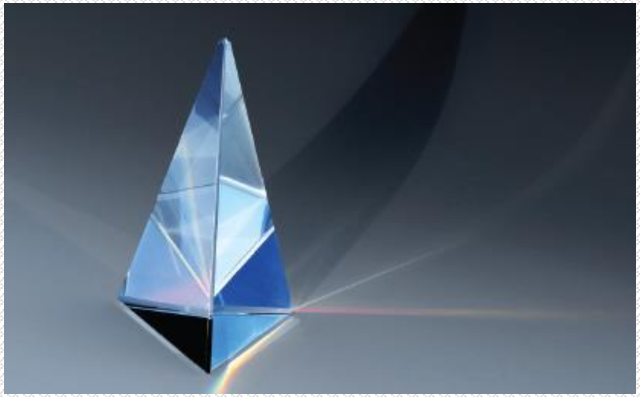
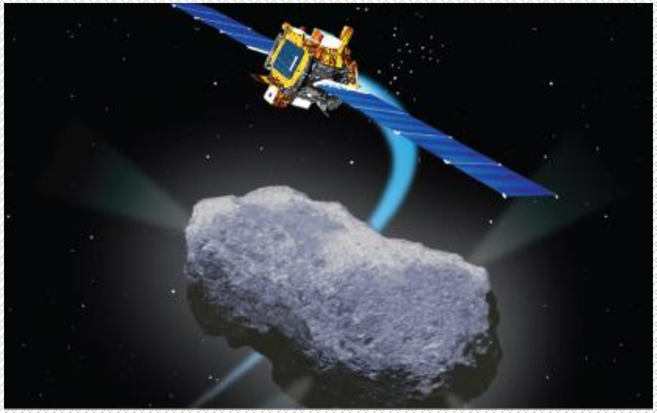
CH 3 Linear Algebra Applied



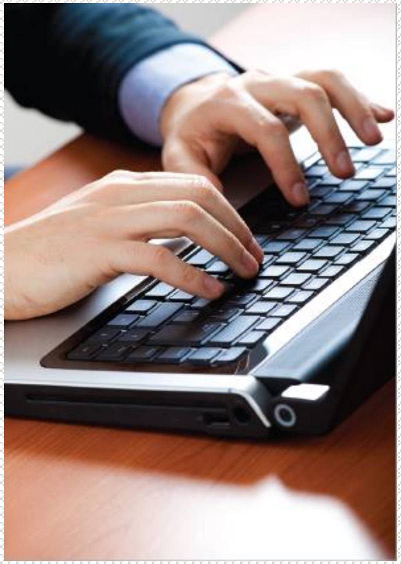
Volume of a Tetrahedron (p.114)

Sudoku (p.120)

Engineering and Control (p.130)



Comet Landing (p.141)



Software Publishing (p.143)

3.1 The Determinant of a Matrix

- the determinant of a 2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\Rightarrow \det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$$

- Note:

$$\left| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

- Ex. 1: (The determinant of a matrix of order 2)

$$\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) = 4 + 3 = 7$$

$$\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2(2) - 4(1) = 4 - 4 = 0$$

$$\begin{vmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{vmatrix} = 0(4) - 2\left(\frac{3}{2}\right) = 0 - 3 = -3$$

- Note: The determinant of a matrix can be positive, zero, or negative.

- Minor of the entry a_{ij}:

The determinant of the matrix determined by deleting the *i*th row and *j*th column of A

$$M_{ij} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & & & \vdots & \vdots & & \\ a_{(i-1)1} & & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & & & \vdots & \vdots & & \vdots \\ a_{n1} & & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}$$

- Cofactor of a_{ij}:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

- Ex:

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Rightarrow M_{21}=\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \qquad M_{22}=\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\Rightarrow C_{21}=(-1)^{2+1}M_{21}=-M_{21} \qquad C_{22}=(-1)^{2+2}M_{22}=M_{22}$$

- Notes: Sign pattern for cofactors

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

3×3 matrix $\qquad$ 4×4 matrix $\qquad$ $n \times n$ matrix

- Notes:

  Odd positions (where $i+j$ is odd) have negative signs, and even positions (where $i+j$ is even) have positive signs.

- Ex 2: Find all the minors and cofactors of *A*.

$$A=\begin{bmatrix}0 & 2 & 1\\ 3 & -1 & 2\\ 4 & 0 & 1\end{bmatrix}$$

Sol: (1) All the minors of *A*.

$$\Rightarrow M_{11}=\begin{vmatrix}-1 & 2\\ 0 & 1\end{vmatrix}=-1,\ M_{12}=\begin{vmatrix}3 & 2\\ 4 & 1\end{vmatrix}=-5,\ M_{13}=\begin{vmatrix}3 & -1\\ 4 & 0\end{vmatrix}=4$$

$$M_{21}=\begin{vmatrix}2 & 1\\ 0 & 1\end{vmatrix}=2,\quad M_{22}=\begin{vmatrix}0 & 1\\ 4 & 1\end{vmatrix}=-4,\ M_{23}=\begin{vmatrix}0 & 2\\ 4 & 1\end{vmatrix}=-8$$

$$M_{31}=\begin{vmatrix}2 & 1\\ -1 & 2\end{vmatrix}=5,\quad M_{32}=\begin{vmatrix}0 & 1\\ 3 & 2\end{vmatrix}=-3,\ M_{33}=\begin{vmatrix}0 & 2\\ 3 & -1\end{vmatrix}=-6$$

Sol: (2) All the cofactors of A.

$$\because \quad C_{ij} = (-1)^{i+j} M_{ij}$$

$$\Rightarrow C_{11} = +\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1, \; C_{12} = -\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 5, \quad C_{13} = +\begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4$$

$$C_{21} = -\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -2, \quad C_{22} = +\begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = -4, \quad C_{23} = -\begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = 8$$

$$C_{31} = +\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5, \quad C_{32} = -\begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = 3, \quad C_{33} = +\begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = -6$$

- Thm 3.1: (Expansion by cofactors)

Let A is a square matrix of order n.

Then the determinant of A is given by

$$(a) \quad \det(A) = |A| = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

(Cofactor expansion along the i-th row, $i = 1, 2, \ldots, n$)

or

$$(b) \quad \det(A) = |A| = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

(Cofactor expansion along the j-th row, $j = 1, 2, \ldots, n$)

- Ex: The determinant of a matrix of order 3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\ &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\ &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} \end{aligned}$$

- Ex 3: The determinant of a matrix of order 3

$$A=\begin{bmatrix}0&2&1\\3&-1&2\\4&0&1\end{bmatrix}\quad \overset{Ex\,2}{\Rightarrow} \begin{array}{l} C_{11}=-1,\ C_{12}=5,\quad C_{13}=4 \\ C_{21}=-2,\ C_{22}=-4,\ C_{23}=8 \\ C_{31}=5,\quad C_{32}=3,\quad C_{33}=-6 \end{array}$$

Sol:

$$\begin{aligned}\Rightarrow \det(A) &= a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}=(0)(-1)+(2)(5)+(1)(4)=14\\ &= a_{21}C_{21}+a_{22}C_{22}+a_{23}C_{23}=(3)(-2)+(-1)(-4)+(2)(8)=14\\ &= a_{31}C_{31}+a_{32}C_{32}+a_{33}C_{33}=(4)(5)+(0)(3)+(1)(-6)=14\\ &= a_{11}C_{11}+a_{21}C_{21}+a_{31}C_{31}=(0)(-1)+(3)(-2)+(4)(5)=14\\ &= a_{12}C_{12}+a_{22}C_{22}+a_{32}C_{32}=(2)(5)+(-1)(-4)+(0)(3)=14\\ &= a_{13}C_{13}+a_{23}C_{23}+a_{33}C_{33}=(1)(4)+(2)(8)+(1)(-6)=14\end{aligned}$$

- Ex 5: (The determinant of a matrix of order 3)

$$A=\begin{bmatrix}0 & 2 & 1\\ 3 & -1 & 2\\ 4 & -4 & 1\end{bmatrix} \quad \Rightarrow \det(A)=?$$

Sol:

$$C_{11}=(-1)^{1+1}\begin{vmatrix}-1 & 2\\ -4 & 1\end{vmatrix}=7 \qquad C_{12}=(-1)^{1+2}\begin{vmatrix}3 & 2\\ 4 & 1\end{vmatrix}=(-1)(-5)=5$$

$$C_{13}=(-1)^{1+3}\begin{vmatrix}3 & -1\\ 4 & -4\end{vmatrix}=-8$$

$$\begin{aligned}\Rightarrow \det(A) &= a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}\\ &= (0)(7)+(2)(5)+(1)(-8)\\ &= 2\end{aligned}$$

- Notes:

  The row (or column) containing the most zeros is the best choice for expansion by cofactors .

- Ex 4: (The determinant of a matrix of order 4)

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix} \Rightarrow \det(A) = ?$$

Sol:

$$\det(A) = (3)(C_{13}) + (0)(C_{23}) + (0)(C_{33}) + (0)(C_{43})$$

$$= 3C_{13}$$

$$= 3(-1)^{1+3}\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix}$$

$$= 3\left[(0)(-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + (2)(-1)^{2+2}\begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} + (3)(-1)^{2+3}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix}\right]$$

$$= 3[0 + (2)(1)(-4) + (3)(-1)(-7)]$$

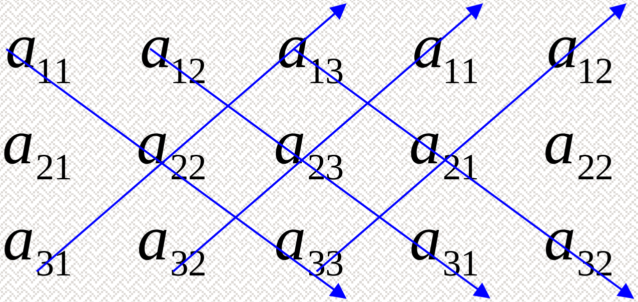
$$= (3)(13)$$

$$= 39$$

- The determinant of a matrix of order 3:

Subtract these three products.

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



Add these three products.

$$\Rightarrow \det(A)=|A|=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{31}a_{22}a_{13}$$
$$-a_{32}a_{23}a_{11}-a_{33}a_{21}a_{12}$$

- Ex 5:

$$\begin{matrix} & & & & -4 & 0 & 6 \\ A = & \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix} & \begin{matrix} 0 & 2 \\ 3 & -1 \\ 4 & -4 \end{matrix} \\ & & & & 0 & 16 & -12 \end{matrix}$$

$$\Rightarrow \det(A) = |A| = 0 + 16 - 12 - (-4) - 0 - 6 = 2$$

- Upper triangular matrix:

  All the entries below the main diagonal are zeros.

- Lower triangular matrix:

  All the entries above the main diagonal are zeros.

- Diagonal matrix:

  All the entries above and below the main diagonal are zeros.

- Note:

  A matrix that is both upper and lower triangular is called diagonal.

- Ex:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

upper triangular　　lower triangular　　diagonal

- Thm 3.2: (Determinant of a Triangular Matrix)

If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then its determinant is the product of the entries on the main diagonal. That is

$$\det(A) = |A| = a_{11}a_{22}a_{33}\cdots a_{nn}$$

- Ex 6: Find the determinants of the following triangular matrices.

$$(a)\quad A=\begin{bmatrix}2 & 0 & 0 & 0\\ 4 & -2 & 0 & 0\\ -5 & 6 & 1 & 0\\ 1 & 5 & 3 & 3\end{bmatrix}\qquad (b)\quad B=\begin{bmatrix}-1 & 0 & 0 & 0 & 0\\ 0 & 3 & 0 & 0 & 0\\ 0 & 0 & 2 & 0 & 0\\ 0 & 0 & 0 & 4 & 0\\ 0 & 0 & 0 & 0 & -2\end{bmatrix}$$

Sol:

$$(a)\quad |A| = (2)(-2)(1)(3) = -12$$

$$(b)\quad |B| = (-1)(3)(2)(4)(-2) = 48$$

老師，這個例題只有(a)小題，請確認是否需要修改

Key Learning in Section 3.1

- Find the determinant of a 2×2 matrix.
- Find the minors and cofactors of a matrix.
- Use expansion by cofactors to find the determinant of a matrix.
- Find the determinant of a triangular matrix.

Keywords in Section 3.1

- determinant : 行列式
- minor : 子行列式
- cofactor : 餘因子
- expansion by cofactors : 餘因子展開
- upper triangular matrix: 上三角矩陣
- lower triangular matrix: 下三角矩陣
- diagonal matrix: 對角矩陣

3.2 Evaluation of a determinant using elementary operations

- Thm 3.3: (Elementary row operations and determinants)

  Let A and B be square matrices.

$$(a)\quad B = r_{ij}(A) \quad \Rightarrow \quad \det(B) = -\det(A) \qquad (\text{i.e. } |r_{ij}(A)| = -|A|)$$

$$(b)\quad B = r_{i}^{(k)}(A) \quad \Rightarrow \quad \det(B) = k\det(A) \qquad (\text{i.e. } |r_{i}^{(k)}(A)| = k|A|)$$

$$(c)\quad B = r_{ij}^{(k)}(A) \quad \Rightarrow \quad \det(B) = \det(A) \qquad (\text{i.e. } |r_{ij}^{(k)}(A)| = |A|)$$

- Ex:

$$A=\begin{bmatrix}1&2&3\\0&1&4\\1&2&1\end{bmatrix}\qquad \det(A)=-2$$

$$A_1=\begin{bmatrix}4&8&12\\0&1&4\\1&2&1\end{bmatrix}\quad A_2=\begin{bmatrix}0&1&4\\1&2&3\\1&2&1\end{bmatrix}\quad A_3=\begin{bmatrix}1&2&3\\-2&-3&-2\\1&2&1\end{bmatrix}$$

$$A_1=r_1^{(4)}(A)\quad\Rightarrow \det(A_1)=\det(r_1^{(4)}(A))=4\det(A)=(4)(-2)=-8$$

$$A_2=r_{12}(A)\quad\Rightarrow \det(A_2)=\det(r_{12}(A))=-\det(A)=-(-2)=2$$

$$A_3=r_{12}^{(-2)}(A)\quad\Rightarrow \det(A_3)=\det(r_{12}^{(-2)}(A))=\det(A)=-2$$

- Notes:

$$\det(r_{ij}(A)) = -\det(A) \quad \Rightarrow \quad \det(A) = -\det(r_{ij}(A))$$

$$\det(r_i^{(k)}(A)) = k\det(A) \quad \Rightarrow \quad \det(A) = \frac{1}{k}\det(r_i^{(k)}(A))$$

$$\det(r_{ij}^{(k)}(A)) = \det(A) \quad \Rightarrow \quad \det(A) = \det(r_{ij}^{(k)}(A))$$

Note:

A row-echelon form of a square matrix is always upper triangular.

- Ex 2: (Evaluation a determinant using elementary row operations)

$$A=\begin{bmatrix}0 & -7 & 14\\ 1 & 2 & -2\\ 0 & 3 & -8\end{bmatrix} \Rightarrow \det(A)=?$$

Sol:

$$\det(A)=\begin{vmatrix}0 & -7 & 14\\ 1 & 2 & -2\\ 0 & 3 & -8\end{vmatrix} \overset{r_{12}}{=} -\begin{vmatrix}1 & 2 & -2\\ 0 & -7 & 14\\ 0 & 3 & -8\end{vmatrix}$$

$$\overset{r_2^{(-\frac{1}{7})}}{=} (-1)\left(\frac{1}{\frac{-1}{7}}\right)\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{vmatrix}$$

$$\overset{r_{23}^{(-3)}}{=} (7)\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & -2 \end{vmatrix} = (7)(1)(1)(-2) = -14$$

- Notes:

$$|EA| = |E||A|$$

(1) $E = R_{ij}$ $\Rightarrow |E| = |R_{ij}| = -1$

$$\Rightarrow |EA| = |r_{ij}(A)| = -|A| = |R_{ij}||A| = |E||A|$$

(2) $E = R_i^{(k)}$ $\Rightarrow |E| = |R_i^{(k)}| = k$

$$\Rightarrow |EA| = |r_i^{(k)}(A)| = k|A| = |R_i^{(k)}||A| = |E||A|$$

(3) $E = R_{ij}^{(k)}$ $\Rightarrow |E| = |R_{ij}^{(k)}| = 1$

$$\Rightarrow |EA| = |r_{ij}^{(k)}(A)| = 1|A| = |R_{ij}^{(k)}||A| = |E||A|$$

- Determinants and elementary column operations

- Thm: (Elementary column operations and determinants)

  Let A and B be square matrices.

  $$(a)\quad B = c_{ij}(A) \quad\Rightarrow\quad \det(B) = -\det(A) \qquad (\text{i.e. } |c_{ij}(A)| = -|A|)$$

  $$(b)\quad B = c_{i}^{(k)}(A) \quad\Rightarrow\quad \det(B) = k\det(A) \qquad (\text{i.e. } |c_{i}^{(k)}(A)| = k|A|)$$

  $$(c)\quad B = c_{ij}^{(k)}(A) \quad\Rightarrow\quad \det(B) = \det(A) \qquad (\text{i.e. } |c_{ij}^{(k)}(A)| = |A|)$$

- Ex:

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \qquad \det(A) = -8$$

$$A_1 = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A_1 = c_1^{(\frac{1}{2})}(A) \quad \Rightarrow \det(A_1) = \det(c_1^{(4)}(A)) = \frac{1}{2}\det(A) = (\frac{1}{2})(-8) = -4$$

$$A_2 = c_{12}(A) \quad \Rightarrow \det(A_2) = \det(c_{12}(A)) = -\det(A) = -(-8) = 8$$

$$A_3 = c_{23}^{(3)}(A) \quad \Rightarrow \det(A_3) = \det(c_{23}^{(3)}(A)) = \det(A) = -8$$

- Thm 3.4: (Conditions that yield a zero determinant)

If A is a square matrix and any of the following conditions is true, then $\det(A) = 0$.

(a) An entire row (or an entire column) consists of zeros.

(b) Two rows (or two columns) are equal.

(c) One row (or column) is a multiple of another row (or column).

- Ex:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 4 & 2 \\ 1 & 5 & 2 \\ 1 & 6 & 2 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -2 & -4 & -6 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 8 & 4 \\ 2 & 10 & 5 \\ 3 & 12 & 6 \end{vmatrix} = 0$$

- Note:

Order n	Cofactor Expansion		Row Reduction	
	Additions	Multiplications	Additions	Multiplications
3	5	9	5	10
5	119	205	30	45
10	3,628,799	6,235,300	285	339

- Ex 5: (Evaluating a determinant)

$$A=\begin{bmatrix}-3 & 5 & 2\\ 2 & -4 & -1\\ -3 & 0 & 6\end{bmatrix}$$

Sol:

$$\det(A)=\begin{vmatrix}-3 & 5 & 2\\ 2 & -4 & -1\\ -3 & 0 & 6\end{vmatrix}\overset{C_{13}^{(2)}}{=}\begin{vmatrix}-3 & 5 & -4\\ 2 & -4 & 3\\ -3 & 0 & 0\end{vmatrix}$$

$$=(-3)(-1)^{3+1}\begin{vmatrix}5 & -4\\ -4 & 3\end{vmatrix}=(-3)(-1)=3$$

$$\det(A)=\begin{vmatrix}-3 & 5 & 2\\ 2 & -4 & -1\\ -3 & 0 & 6\end{vmatrix}\overset{r_{12}^{(\frac{4}{5})}}{=}\begin{vmatrix}-3 & 5 & 2\\ \frac{-2}{5} & 0 & \frac{3}{5}\\ -3 & 0 & 6\end{vmatrix}$$

$$=(5)(-1)^{1+2}\begin{vmatrix}\frac{-2}{5} & \frac{3}{5}\\ -3 & 6\end{vmatrix}=(-5)(-\frac{3}{5})=3$$

- Ex 6: (Evaluating a determinant)

$$A=\begin{bmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix}$$

Sol:

$$\det(A)=\begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{vmatrix} \underset{\substack{r_{24}^{(1)} \\ r_{25}^{(-1)}}}{=} \begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 1 & 0 & 5 & 6 & -4 \\ 3 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$=(1)(-1)^{2+2}\begin{vmatrix} 2 & 1 & 3 & -2 \\ 1 & -1 & 2 & 3 \\ 1 & 5 & 6 & -4 \\ 3 & 0 & 0 & 1 \end{vmatrix}$$

$$\underset{C_{41}^{(-3)}}{=}\begin{vmatrix} 8 & 1 & 3 & -2 \\ -8 & -1 & 2 & 3 \\ 13 & 5 & 6 & -4 \\ 0 & 0 & 0 & 1 \end{vmatrix} = (1)(-1)^{4+4}\begin{vmatrix} 8 & 1 & 3 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} \underset{r_{21}^{(1)}}{=} \begin{vmatrix} 0 & 0 & 5 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix}$$

$$= 5(-1)^{1+3}\begin{vmatrix} -8 & -1 \\ 13 & 5 \end{vmatrix}$$

$$= (5)(-27)$$

$$= -135$$

Key Learning in Section 3.2

- Use elementary row operations to evaluate a determinant.
- Use elementary column operations to evaluate a determinant.
- Recognize conditions that yield zero determinants.

Keywords in Section 3.2

- determinant : 行列式
- elementary row operation: 基本列運算
- row equivalent: 列等價
- elementary matrix: 基本矩陣
- elementary column operation: 基本行運算
- column equivalent: 行等價

3.3 Properties of Determinants

- Thm 3.5: (Determinant of a matrix product)

$$\det(AB) = \det(A)\det(B)$$

- Notes:

(1) $\det(EA) = \det(E)\det(A)$

(2) $\det(A+B) \neq \det(A) + \det(B)$

(3)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{22}+b_{22} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- Ex 1: (The determinant of a matrix product)

$$A=\begin{bmatrix}1 & -2 & 2\\ 0 & 3 & 2\\ 1 & 0 & 1\end{bmatrix} \qquad B=\begin{bmatrix}2 & 0 & 1\\ 0 & -1 & -2\\ 3 & 1 & -2\end{bmatrix}$$

Find $|A|$, $|B|$, and $|AB|$

Sol:

$$|A|=\begin{vmatrix}1 & -2 & 2\\ 0 & 3 & 2\\ 1 & 0 & 1\end{vmatrix}=-7 \qquad |B|=\begin{vmatrix}2 & 0 & 1\\ 0 & -1 & -2\\ 3 & 1 & -2\end{vmatrix}=11$$

$$AB = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{bmatrix}$$

$$\Rightarrow |AB| = \begin{vmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{vmatrix} = -77$$

- Check:

$$|AB| = |A|\ |B|$$

- Thm 3.6: (Determinant of a scalar multiple of a matrix)

If A is an $n \times n$ matrix and c is a scalar, then

$$\det(cA) = c^n \det(A)$$

- Ex 2:

$$A = \begin{bmatrix} 10 & -20 & 40 \\ 30 & 0 & 50 \\ -20 & -30 & 10 \end{bmatrix}, \quad \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = 5$$

Find $|A|$.

Sol:

$$A = 10\begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{bmatrix} \Rightarrow |A| = 10^3 \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = (1000)(5) = 5000$$

- **Thm 3.7: (Determinant of an invertible matrix)**

A square matrix A is invertible (nonsingular) if and only if

$$\det(A) \neq 0$$

- **Ex 3: (Classifying square matrices as singular or nonsingular)**

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

Sol:

$|A| = 0 \quad \Rightarrow$ A has no inverse (it is singular).

$|B| = -12 \neq 0 \quad \Rightarrow$ B has an inverse (it is nonsingular).

- Thm 3.8: (Determinant of an inverse matrix)

If A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

- Thm 3.9: (Determinant of a transpose)

If A is a square matrix, then $\det(A^{\mathrm{T}}) = \det(A)$.

- Ex 4:

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \qquad \text{(a) } \left|A^{-1}\right| = ? \qquad \text{(b) } \left|A^{T}\right| = ?$$

Sol:

$$\because \quad |A| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{vmatrix} = 4$$

$$\therefore \quad \left|A^{-1}\right| = \frac{1}{|A|} = \frac{1}{4}$$

$$\left|A^{T}\right| = |A| = 4$$

- Equivalent conditions for a nonsingular matrix:

If A is an $n \times n$ matrix, then the following statements are equivalent.

(1) A is invertible.

(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$.

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(4) A is row-equivalent to I_n

(5) A can be written as the product of elementary matrices.

(6) $\det(A) \neq 0$

- Ex 5: Which of the following system has a unique solution?

$$(a)\quad \begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= -4 \end{aligned}$$

$$(b)\quad \begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + x_3 &= -4 \end{aligned}$$

Sol:

$(a)\quad A\mathbf{x} = \mathbf{b}$

$\because \quad |A| = 0$

$\therefore$ This system does not have a unique solution.

$(b)\quad B\mathbf{x} = \mathbf{b}$

$\because \quad |B| = -12 \neq 0$

$\therefore$ This system has a unique solution.

Key Learning in Section 3.3

- Find the determinant of a matrix product and a scalar multiple of a matrix.
- Find the determinant of an inverse matrix and recognize equivalent conditions for a nonsingular matrix.
- Find the determinant of the transpose of a matrix.

Keywords in Section 3.3

- determinant: 行列式
- matrix multiplication: 矩陣相乘
- scalar multiplication: 純量積
- invertible matrix: 可逆矩陣
- inverse matrix: 反矩陣
- nonsingular matrix: 非奇異矩陣
- transpose matrix: 轉置矩陣

3.4 Applications of Determinants

- Matrix of cofactors of A:

$$[C_{ij}] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} \qquad C_{ij} = (-1)^{i+j} M_{ij}$$

- Adjoint matrix of A:

$$adj(A) = [C_{ij}]^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

- Thm 3.10: (The inverse of a matrix given by its adjoint)

  If A is an $n \times n$ invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

- Ex:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \det(A) = ad - bc$$

$$adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{\det(A)} adj(A)$$

$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Ex 1 & Ex 2:

$$A=\begin{bmatrix}-1 & 3 & 2\\ 0 & -2 & 1\\ 1 & 0 & -2\end{bmatrix}$$

(a) Find the adjoint of A.

(b) Use the adjoint of A to find A^{-1}

Sol: $\because \quad C_{ij}=(-1)^{i+j}M_{ij}$

$$\Rightarrow C_{11}=+\begin{vmatrix}-2 & 1\\ 0 & -2\end{vmatrix}=4,\ C_{12}=-\begin{vmatrix}0 & 1\\ 1 & -2\end{vmatrix}=1,\ C_{13}=+\begin{vmatrix}0 & -2\\ 1 & 0\end{vmatrix}=2$$

$$C_{21}=-\begin{vmatrix}3 & 2\\ 0 & 2\end{vmatrix}=6,\quad C_{22}=+\begin{vmatrix}-1 & 2\\ 1 & 2\end{vmatrix}=0,\ C_{23}=-\begin{vmatrix}-1 & 3\\ 1 & 0\end{vmatrix}=3$$

$$C_{31}=+\begin{vmatrix}3 & 2\\ -2 & 1\end{vmatrix}=7,\quad C_{32}=-\begin{vmatrix}-1 & 2\\ 0 & 1\end{vmatrix}=1,\ C_{33}=+\begin{vmatrix}-1 & 3\\ 0 & -2\end{vmatrix}=2$$

$\Rightarrow$ cofactor matrix of A

$$[C_{ij}] = \begin{bmatrix} 4 & 1 & 2 \\ 6 & 0 & 3 \\ 7 & 1 & 2 \end{bmatrix}$$

$\Rightarrow$ adjoint matrix of A

$$adj(A) = [C_{ij}]^T = \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

$\Rightarrow$ inverse matrix of A

$$A^{-1} = \frac{1}{\det(A)} adj(A) \qquad \because \quad \det(A) = 3$$

$$= \begin{bmatrix} \frac{4}{3} & 2 & \frac{7}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} \end{bmatrix}$$

- Check: $AA^{-1} = I$

- Cramer's Rule

Cramer's Rule uses determinants to solve a system of linear equations in variables. This rule applies only to systems with unique solutions.

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \qquad x_2 = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{21}a_{12}} \qquad a_{11}a_{22} - a_{21}a_{12} \neq 0$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \qquad |A_1| = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \qquad |A_2| = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

$$x_1 = \frac{|A_1|}{|A|} \qquad x_2 = \frac{|A_2|}{|A|}$$

- Ex 3: (Using Cramer's Rule)

Use Cramer's Rule to solve the system of linear equations.

$$\begin{aligned} 4x_1 - 2x_2 &= 10 \\ 3x_1 - 5x_2 &= 11 \end{aligned}$$

Sol: Find the determinant of the coefficient matrix

$$|A| = \begin{vmatrix} 4 & -2 \\ 3 & -5 \end{vmatrix} = -14$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 10 & -2 \\ 11 & -5 \end{vmatrix}}{-14} = \frac{-28}{-14} = 2$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 4 & 10 \\ 3 & 11 \end{vmatrix}}{-14} = \frac{14}{-14} = -1$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

$$x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

- Thm 3.11: (Cramer's Rule)

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

$$A\mathbf{x} = \mathbf{b} \qquad A = \left[a_{ij}\right]_{n\times n} = \left[A^{(1)}, A^{(2)}, \cdots, A^{(n)}\right] \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

(this system has a unique solution)

$$A_j = \left[A^{(1)}, A^{(2)}, \cdots, A^{(j-1)}, b, A^{(j+1)}, \cdots, A^{(n)}\right]$$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2(j-1)} & b_2 & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & & & \ddots & & & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & b_n & a_{n(j+1)} & \cdots & a_{nn} \end{bmatrix}$$

$$(\text{ i.e. } \det(A_j) = b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj})$$

$$\Rightarrow x_j = \frac{\det(A_j)}{\det(A)}, \qquad j = 1, 2, \cdots, n$$

Pf:

$$A\mathbf{x} = \mathbf{b}, \qquad \det(A) \neq 0$$

$$\Rightarrow \mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det(A)} adj(A)\mathbf{b}$$

$$= \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \frac{1}{\det(A)} \begin{bmatrix} b_1C_{11} & + & b_2C_{21} & + & \cdots & + & b_nC_{n1} \\ b_1C_{12} & + & b_2C_{22} & + & \cdots & + & b_nC_{n2} \\ & \vdots & & \vdots & & & \\ b_1C_{1n} & + & b_2C_{2n} & + & \cdots & + & b_nC_{nn} \end{bmatrix}$$

$$\Rightarrow x_j = \frac{1}{\det(A)}(b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj})$$

$$= \frac{\det(A_j)}{\det(A)} \qquad j = 1, 2, \cdots, n$$

- Ex 4: Use Cramer's rule to solve the system of linear equations.

$$\begin{aligned} -x + 2y - 3z &= 1 \\ 2x + z &= 0 \\ 3x - 4y + 4z &= 2 \end{aligned}$$

Sol:

$$\det(A) = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} = 10 \qquad \det(A_1) = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix} = 8$$

$$\det(A_2) = \begin{vmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix} = -15, \qquad \det(A_3) = \begin{vmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{vmatrix} = -16$$

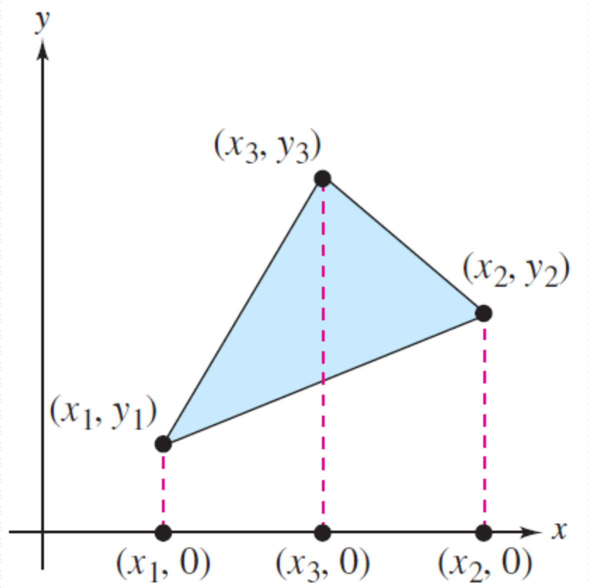
$$x = \frac{\det(A_1)}{\det(A)} = \frac{4}{5} \qquad y = \frac{\det(A_2)}{\det(A)} = \frac{-3}{2} \qquad z = \frac{\det(A_3)}{\det(A)} = \frac{-8}{5}$$

- Area of a triangle in the *xy*-plane:

A triangle with vertices

$(x_1, y_1), (x_2, y_2)$, and (x_3, y_3)

$$\text{Area} = \pm\frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$



where the sign (±) is chosen to give a positive area.

Pf:

Consider the three trapezoid

Trapezoid 1: $(x_1, 0), (x_1, y_1), (x_3, y_3), (x_3, 0)$

Trapezoid 2: $(x_3, 0), (x_3, y_3), (x_2, y_2), (x_2, 0)$

Trapezoid 3: $(x_1, 0), (x_1, y_1), (x_2, y_2), (x_2, 0)$

$$\text{Area} = \frac{1}{2}(y_1 + y_3)(x_3 - x_1) + \frac{1}{2}(y_3 + y_2)(x_2 - x_3) - \frac{1}{2}(y_1 + y_2)(x_2 - x_1)$$
$$= \frac{1}{2}(x_1 y_2 + x_2 y_3 + x_3 y_1 - x_1 y_3 - x_2 y_1 - x_3 y_2)$$
$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

If the vertices do not occur in the order $x_1 \leqq x_2 \leqq x_3$ or if the vertex (x_3, y_3) is not above the line segment connecting the other two vertices, then the formula above may yield the negative of the area. So, use $\pm$ and choose the correct sign to give a positive area.

- Ex 5: (Finding the Area of a Triangle)

Find the area of the triangle whose vertices are (1, 1), (2, 2), and (4, 3).

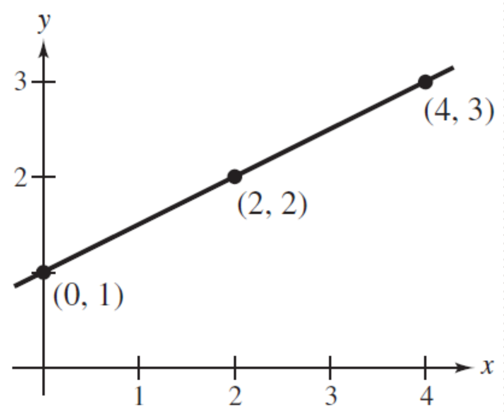
Sol:

$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = -\frac{3}{2}$$

The area of the triangle is $\frac{3}{2}$ square units.

If three points in the *xy*-plane lie on the same line, then the determinant in the formula for the area of a triangle is zero.

$$\frac{1}{2}\begin{vmatrix} 0 & 1 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = 0$$



- Test for collinear points in the *xy*-plane:

Three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear if and only if

$$\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0$$

- Two-point form of the equation of a line:

An equation of the line passing through the distinct points (x_1, y_1) and (x_2, y_2) is given by

$$\det\begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0$$

- Ex 6: (Finding an Equation of the Line Passing Through Two Points)

Find an equation of the line passing through the points (2, 4) and (-1, 3).

Sol:

$$\begin{vmatrix} x & y & 1 \\ 2 & 4 & 1 \\ -1 & 3 & 1 \end{vmatrix} = 0$$

$$x\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} - y\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} = 0$$

$$x(1) - y(3) + 1(10) = 0$$

$$x - 3y = -10$$

- Volume of a Tetrahedron:

The volume of a tetrahedron with vertices (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4) is

$$\text{Volume} = \det\begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix}$$

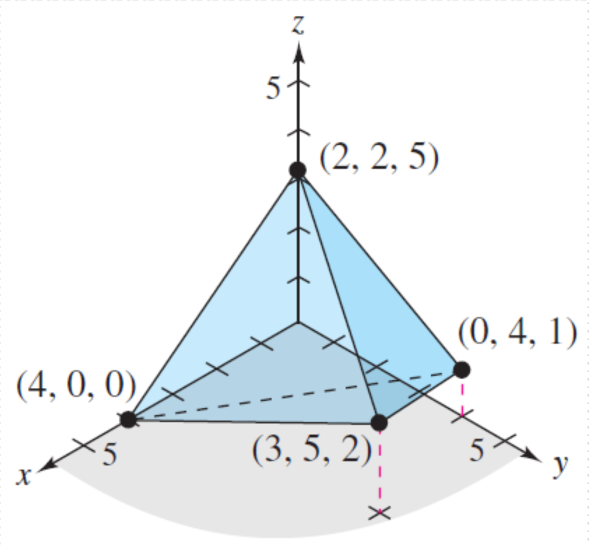
where the sign (±) is chosen to give a positive area.

- Ex 7: (Finding the Volume of a Tetrahedron)

  Find the volume of the tetrahedron shown in the following figure, whose vertices are (0, 4, 1), (4, 0, 0), (3, 5, 2), and (2, 2, 5).

Sol:

$$\frac{1}{2}\begin{vmatrix} 0 & 4 & 1 & 1 \\ 4 & 0 & 0 & 1 \\ 3 & 5 & 2 & 1 \\ 2 & 2 & 5 & 1 \end{vmatrix} = -12$$



The volume of the tetrahedron is 12 cubic units.

- Test for coplanar points in space:

Four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4) are coplanar if and only if

$$\det\begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix} = 0$$

- Three-point form of the equation of a line:

An equation of the line passing through the distinct points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) is given by

$$\det\begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} = 0$$

- Ex 8: (Finding an Equation of the Plane Passing Through Three Points)

  Find an equation of the plane passing through the points (0, 1, 0), (−1, 3, 2), and (-2, 0, 1).

Sol:

$$\begin{vmatrix} x & y & z & 1 \\ 0 & 1 & 0 & 1 \\ -1 & 3 & 2 & 1 \\ -2 & 0 & 1 & 1 \end{vmatrix} = 0 \qquad \begin{vmatrix} x & y-1 & z & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 2 & 2 & 1 \\ -2 & -1 & 1 & 1 \end{vmatrix} = 0$$

$$x\begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix} - (y-1)\begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix} + z\begin{vmatrix} -1 & 2 \\ -2 & -1 \end{vmatrix} = 0$$

$$x(4) - (y-1)(3) + z(5) = 0$$

$$4x - 3y + 5z = -3$$

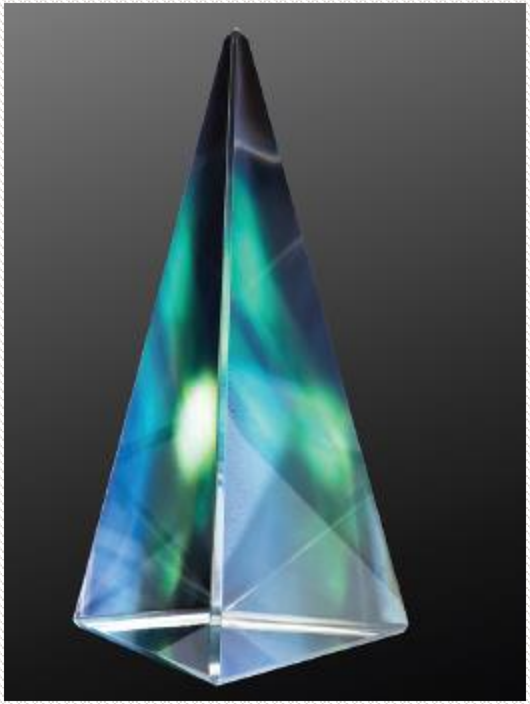
Key Learning in Section 3.4

- Find the adjoint of a matrix and use it to find the inverse of the matrix.
- Use Cramer's Rule to solve a system of n linear equations in n variables.
- Use determinants to find area, volume, and the equations of lines and planes.

Keywords in Section 3.4

- matrix of cofactors : 餘因子矩陣
- adjoint matrix : 伴隨矩陣
- Cramer's rule : 克萊姆法則
- area: 面積
- volume: 體積
- triangle: 三角形
- tetrahedron: 四面體

- Volume of a Tetrahedron



Recall that a **tetrahedron** is a polyhedron consisting of four triangular faces. One practical application of determinants is in finding the volume of a tetrahedron in a coordinate plane. If the vertices of a tetrahedron are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4), then the volume is

$$\text{Volume} = \pm\tfrac{1}{6}\det\begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix}$$

You will study this and other applications of determinants in Section 3.4.

- Sudoku

In a Sudoku puzzle, the object is to fill out a partially completed 9×9 grid of boxes with numbers from 1 to 9 so that each column, row, and 3×3 sub-grid contains each of these numbers without repetition. For a completed Sudoku grid to be valid, no two rows (or columns) will have the numbers in the same order. If this should happen, then the determinant of the matrix formed by the numbers will be zero. This is a direct result of condition 2 of Theorem 3.4.

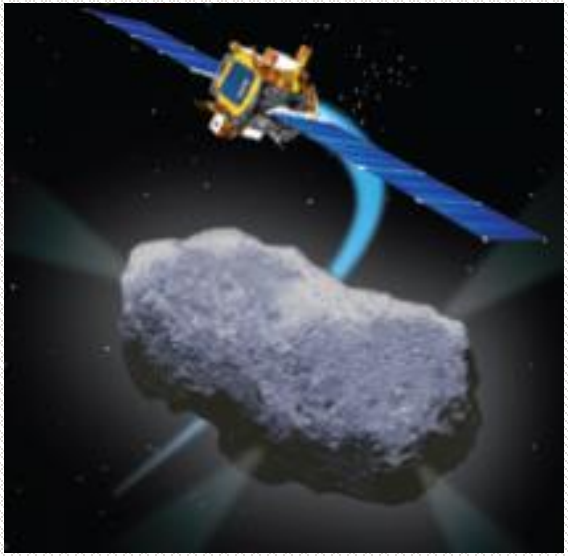
- Engineering and Control

Systems of linear differential equations often arise in engineering and control theory. For a function $f(t)$ that is defined for all positive values of t, the **Laplace transform** of $f(t)$ is

$$F(s) = \int_0^\infty e^{-st} f(t)dt.$$

provided that the improper integral exists. Laplace transforms and Cramer's Rule, which uses determinants to solve a system of linear equations, can sometimes be used to solve a system of differential equations. You will study Cramer's Rule in the next section.

Comet Landing



On November 12, 2014, European Space Agency's Rosetta orbiting spacecraft landed the probe Philae on the surface of the comet 67P/Churyumov-Gerasimenko. Comets that orbit the Sun, such as 67P, follow Kepler's First Law of Planetary Motion. This law states that the orbit is an ellipse, with the sun at one focus of the ellipse. The general equation of a conic section, such as an ellipse, is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

To determine the equation of the comet's orbit, astronomers can find the coordinates of the comet at five different points (x_i, y_i) where $i = 1, 2, 3, 4,$ and 5, substitute these coordinates into the equation

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix} = 0$$

and then expand by cofactors in the first row to find a, b, c, d, e, and f. For example, the coefficient of x^2 is

$$a = \begin{vmatrix} x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix}$$

Knowing the equation of 67P's orbit helped astronomers determine the ideal time to release the probe.