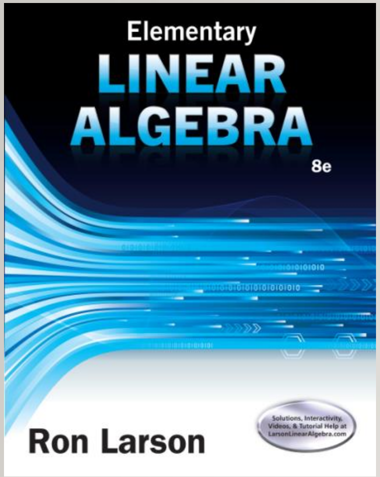

CHAPTER 4
VECTOR SPACES

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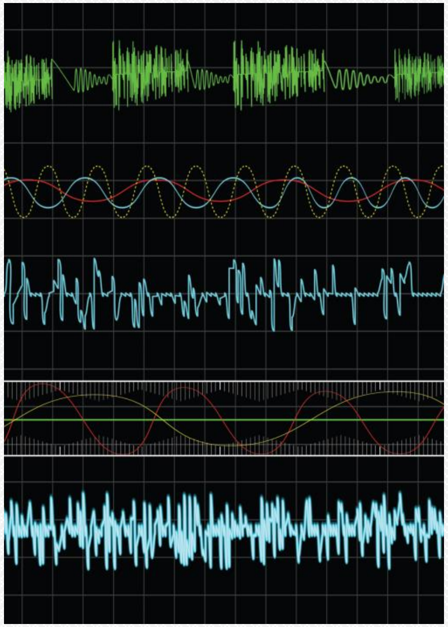
CH 4 Linear Algebra Applied

Force (p.157)



Digital Sampling (p.172)

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4.1 Vectors in R^n

- An ordered n-tuple:

  a sequence of n real number $(x_1, x_2, \cdots, x_n)$

- n-space: R^n

  the set of all ordered n-tuple

- Ex:

$n = 1$ R^1 = 1-space

= set of all real number

$n = 2$ R^2 = 2-space

= set of all ordered pair of real numbers (x_1, x_2)

$n = 3$ R^3 = 3-space

= set of all ordered triple of real numbers (x_1, x_2, x_3)

$n = 4$ R^4 = 4-space

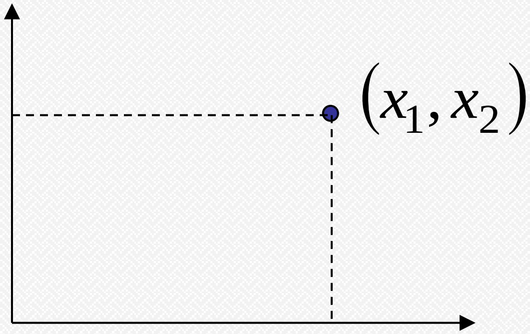
= set of all ordered quadruple of real numbers (x_1, x_2, x_3, x_4)

- **Notes:**

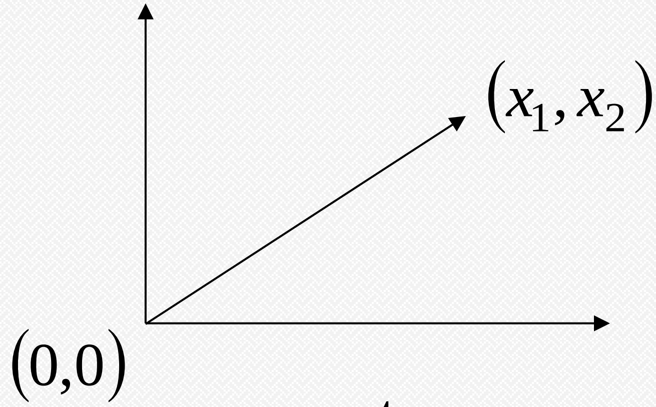
(1) An n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as a point in R^n with the x_i's as its coordinates.

(2) An n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as a vector $x = (x_1, x_2, \cdots, x_n)$ in R^n with the x_i's as its components.

- **Ex:**



a point



a vector

$$\mathbf{u}=(u_1,u_2,\cdots,u_n),\ \mathbf{v}=(v_1,v_2,\cdots,v_n) \qquad \text{(two vectors in } R^n\text{)}$$

- Equal:

$$\mathbf{u}=\mathbf{v} \quad \text{if and only if} \quad u_1=v_1,\ u_2=v_2,\cdots,u_n=v_n$$

- Vector addition (the sum of **u** and **v**):

$$\mathbf{u}+\mathbf{v}=(u_1+v_1,u_2+v_2,\cdots,u_n+v_n)$$

- Scalar multiplication (the scalar multiple of **u** by c):

$$c\mathbf{u}=(cu_1,cu_2,\cdots,cu_n)$$

- Notes:

The sum of two vectors and the scalar multiple of a vector in R^n are called the standard operations in R^n.

- Negative:

$$-\mathbf{u} = (-u_1, -u_2, -u_3, \ldots, -u_n)$$

- Difference:

$$\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2, u_3 - v_3, \ldots, u_n - v_n)$$

- Zero vector:

$$\mathbf{0} = (0, 0, \ldots, 0)$$

- Notes:

(1) The zero vector $\mathbf{0}$ in R^n is called the **additive identity** in R^n.

(2) The vector $-\mathbf{v}$ is called the **additive inverse** of $\mathbf{v}$.

- Thm 4.2: (Properties of vector addition and scalar multiplication)

Let **u**, **v**, and **w** be vectors in R^n , and let c and d be scalars.

(1) $\mathbf{u}+\mathbf{v}$ is a vector in R^n

(2) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$

(3) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$

(4) $\mathbf{u}+\mathbf{0} = \mathbf{u}$

(5) $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$

(6) $c\mathbf{u}$ is a vector in R^n

(7) $\mathrm{c}(\mathbf{u}+\mathbf{v}) = \mathrm{c}\mathbf{u}+\mathrm{c}\mathbf{v}$

(8) $(c+d)\mathbf{u} = \mathrm{c}\mathbf{u}+\mathrm{d}\mathbf{u}$

(9) $c(\mathrm{d}\mathbf{u}) = (\mathrm{cd})\mathbf{u}$

(10) $1(\mathbf{u}) = \mathbf{u}$

- Ex 5: (Vector operations in R^4)

Let $\mathbf{u}=(2, -1, 5, 0)$, $\mathbf{v}=(4, 3, 1, -1)$, and $\mathbf{w}=(-6, 2, 0, 3)$ be vectors in R^4. Solve $\mathbf{x}$ for x in each of the following.

(a) $\mathbf{x} = 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w})$

(b) $3(\mathbf{x}+\mathbf{w}) = 2\mathbf{u} - \mathbf{v}+\mathbf{x}$

Sol: (a)

$$\begin{aligned}
\mathbf{x} &= 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w}) \\
&= 2\mathbf{u} - \mathbf{v} - 3\mathbf{w} \\
&= (4, -2, 10, 0) - (4, 3, 1, -1) - (-18, 6, 0, 9) \\
&= (4-4+18, -2-3-6, 10-1-0, 0+1-9) \\
&= (18, -11, 9, -8).
\end{aligned}$$

(b) $3(\mathbf{x}+\mathbf{w}) = 2\mathbf{u}-\mathbf{v}+\mathbf{x}$

$$3\mathbf{x}+3\mathbf{w} = 2\mathbf{u}-\mathbf{v}+\mathbf{x}$$

$$3\mathbf{x}-\mathbf{x} = 2\mathbf{u}-\mathbf{v}-3\mathbf{w}$$

$$2\mathbf{x} = 2\mathbf{u}-\mathbf{v}-3\mathbf{w}$$

$$\mathbf{x} = \mathbf{u}-\tfrac{1}{2}\mathbf{v}-\tfrac{3}{2}\mathbf{w}$$

$$= (2,1,5,0)+\left(-2,\tfrac{-3}{2},\tfrac{-1}{2},\tfrac{1}{2}\right)+\left(9,-3,0,\tfrac{-9}{2}\right)$$

$$= \left(9,\tfrac{-11}{2},\tfrac{9}{2},-4\right)$$

- Thm 4.3: (Properties of additive identity and additive inverse)

  Let **v** be a vector in R^n and c be a scalar. Then the following is true.

(1) The additive identity is unique. That is, if $\mathbf{u}+\mathbf{v}=\mathbf{v}$, then $\mathbf{u} = \mathbf{0}$

(2) The additive inverse of **v** is unique. That is, if $\mathbf{v}+\mathbf{u}=\mathbf{0}$, then $\mathbf{u} = -\mathbf{v}$

(3) $0\mathbf{v}=\mathbf{0}$

(4) $c\mathbf{0}=\mathbf{0}$

(5) If $c\mathbf{v}=\mathbf{0}$, then $c=0$ or $\mathbf{v}=\mathbf{0}$

(6) $-(-\mathbf{v}) = \mathbf{v}$

- Linear combination:

  The vector **x** is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$,

  if it can be expressed in the form

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \qquad c_1, c_2, \cdots, c_n : \text{scalar}$$

- Ex 6:

  Given **x** = (– 1, – 2, – 2), **u** = (0,1,4), **v** = (– 1,1,2), and **w** = (3,1,2) in R^3, find *a*, *b*, and *c* such that **x** = *a***u**+*b***v**+*c***w**.

Sol:

$$\begin{array}{rcrcrcr} & & -b & + & 3c & = & -1 \\ a & + & b & + & c & = & -2 \\ 4a & + & 2b & + & 2c & = & -2 \end{array}$$

$$\Rightarrow a = 1,\ b = -2,\ c = -1$$

Thus $\mathbf{x} = \mathbf{u} - 2\mathbf{v} - \mathbf{w}$

- Notes:

A vector $\mathbf{u}=(u_1,u_2,\ldots,u_n)$ in R^n can be viewed as:

a 1×*n* row matrix (row vector): $\mathbf{u}=[u_1,\ u_2,\cdots,u_n]$

or

a *n*×1 column matrix (column vector): $\mathbf{u}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}$

(The matrix operations of addition and scalar multiplication give the same results as the corresponding vector operations)

Vector addition

$$\mathbf{u}+\mathbf{v}=(u_1,u_2,\cdots,u_n)+(v_1,v_2,\cdots,v_n)$$
$$=(u_1+v_1,u_2+v_2,\cdots,u_n+v_n)$$

$$\mathbf{u}+\mathbf{v}=[u_1,u_2,\cdots,u_n]+[v_1,v_2,\cdots,v_n]$$
$$=[u_1+v_1,u_2+v_2,\cdots,u_n+v_n]$$

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=\begin{bmatrix}u_1+v_1\\u_2+v_2\\\vdots\\u_n+v_n\end{bmatrix}$$

Scalar multiplication

$$c\mathbf{u}=c(u_1,\ u_2,\cdots,\ u_n)$$
$$=(cu_1,cu_2,\cdots,\ cu_n)$$

$$c\mathbf{u}=c[u_1,\ u_2,\cdots,\ u_n]$$
$$=[cu_1,cu_2,\cdots,\ cu_n]$$

$$c\mathbf{u}=c\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}=\begin{bmatrix}cu_1\\cu_2\\\vdots\\cu_n\end{bmatrix}$$

Key Learning in Section 4.1

- Represent a vector as a directed line segment.
- Perform basic vector operations in R^2 and represent them graphically.
- Perform basic vector operations in R^n.
- Write a vector as a linear combination of other vectors.

Keywords in Section 4.1

- ordered *n*-tuple：有序的*n*項
- *n*-space：*n*維空間
- equal：相等
- vector addition：向量加法
- scalar multiplication：純量乘法
- negative：負向量
- difference：向量差
- zero vector：零向量
- additive identity：加法單位元素
- additive inverse：加法反元素

4.2 Vector Spaces

- Vector spaces:

  Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the following axioms are satisfied for every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and every scalar (real number) c and d, then V is called a **vector space**.

Addition:

(1) $\mathbf{u}+\mathbf{v}$ is in V

(2) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(3) $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$

(4) V has a zero vector $\mathbf{0}$ such that for every $\mathbf{u}$ in V, $\mathbf{u}+\mathbf{0}=\mathbf{u}$

(5) For every $\mathbf{u}$ in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$

Scalar multiplication:

(6) $c\mathbf{u}$ is in V.

(7) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(8) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(9) $c(d\mathbf{u})=(cd)\mathbf{u}$

(10) $1(\mathbf{u})=\mathbf{u}$

- Notes:

(1) A vector space consists of four entities:

a set of vectors, a set of scalars, and two operations

V：nonempty set

c：scalar

$+(\mathbf{u},\mathbf{v})=\mathbf{u}+\mathbf{v}:$ vector addition

$\bullet(c,\mathbf{u})=c\mathbf{u}:$ scalar multiplication

$(V, \quad +, \quad \bullet)$ is called a vector space

(2) $V=\{\mathbf{0}\}$: zero vector space

- Examples of vector spaces:

(1) n-tuple space: R^n

$$(u_1,u_2,\cdots,u_n)+(v_1,v_2,\cdots,v_n)=(u_1+v_1,u_2+v_2,\cdots,u_n+v_n)$$ vector addition

$$k(u_1,u_2,\cdots,u_n)=(ku_1,ku_2,\cdots,ku_n)$$ scalar multiplication

(2) Matrix space: $V = M_{m\times n}$ (the set of all $m\times n$ matrices with real values)

Ex: ：$(m = n = 2)$

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}+\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}=\begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$ vector addition

$$k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}=\begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$ scalar multiplication

(3) *n*-th degree polynomial space: $V = P_n(x)$

(the set of all real polynomials of degree *n* or less)

$$p(x)+q(x)=(a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n$$

$$kp(x)=ka_0+ka_1x+\cdots+ka_nx^n$$

(4) Function space: $V = c(-\infty,\infty)$ （the set of all real-valued continuous functions defined on the entire real line.)

$$(f+g)(x)=f(x)+g(x)$$

$$(kf)(x)=kf(x)$$

- Thm 4.4: (Properties of scalar multiplication)

Let $\mathbf{v}$ be any element of a vector space V, and let c be any scalar. Then the following properties are true.

(1) $0\mathbf{v} = \mathbf{0}$

(2) $c\mathbf{0} = \mathbf{0}$

(3) If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$

(4) $(-1)\mathbf{v} = -\mathbf{v}$

- Notes: To show that a set is not a vector space, you need only find one axiom that is not satisfied.

- Ex 6: The set of all integer is not a vector space.

Pf:

$$1 \in V, \tfrac{1}{2} \in R$$

$$\underset{\substack{\uparrow \\ \text{scalar}}}{(\tfrac{1}{2})}\underset{\substack{\uparrow \\ \text{integer}}}{(1)} = \underset{\substack{\uparrow \\ \text{noninteger}}}{\tfrac{1}{2}} \notin V$$

（it is not closed under scalar multiplication）

- Ex 7: The set of all second-degree polynomials is not a vector space.

Pf: Let $p(x) = x^2$ and $q(x) = -x^2 + x + 1$

$$\Rightarrow p(x) + q(x) = x + 1 \notin V$$

(it is not closed under vector addition)

- Ex 8:

$V=R^2$=the set of all ordered pairs of real numbers

vector addition: $(u_1,u_2)+(v_1,v_2)=(u_1+v_1,u_2+v_2)$

scalar multiplication: $c(u_1,u_2)=(cu_1,0)$

Verify V is not a vector space.

Sol:

$\because 1(1,1)=(1,0)\neq(1,1)$

$\therefore$ the set (together with the two given operations) is not a vector space

Key Learning in Section 4.2

- Define a vector space and recognize some important vector spaces.
- Show that a given set is not a vector space.

Keywords in Section 4.2:

- vector space：向量空間
- *n*-space：*n*維空間
- matrix space：矩陣空間
- polynomial space：多項式空間
- function space：函數空間

4.3 Subspaces of Vector Spaces

- Subspace:

$(V,+,\bullet)$: a vector space

$$\left.\begin{array}{l} W \neq \phi \\ W \subseteq V \end{array}\right\}$$: a nonempty subset

$(W,+,\bullet)$: a vector space (under the operations of addition and scalar multiplication defined in V)

$\Rightarrow$ W is a subspace of V

- Trivial subspace:

Every vector space V has at least two subspaces.

(1) Zero vector space $\{\mathbf{0}\}$ is a subspace of V.

(2) V is a subspace of V.

- Thm 4.5: (Test for a subspace)

If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following conditions hold.

(1) If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ is in W.

(2) If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W.

- Ex: Subspace of R^2

  (1) $\{\mathbf{0}\}$ $\mathbf{0} = (0, 0)$

  (2) Lines through the origin

  (3) R^2

- Ex: Subspace of R^3

  (1) $\{\mathbf{0}\}$ $\mathbf{0} = (0, 0, 0)$

  (2) Lines through the origin

  (3) Planes through the origin

  (4) R^3

- Ex 2: (A subspace of $M_{2\times 2}$)

  Let W be the set of all 2×2 symmetric matrices. Show that W is a subspace of the vector space $M_{2\times 2}$, with the standard operations of matrix addition and scalar multiplication.

Sol:

$\because W \subseteq M_{2\times 2} \qquad M_{2\times 2}: \text{vector sapces}$

$\text{Let } A_1, A_2 \in W \ (A_1^T = A_1, A_2^T = A_2)$

$A_1 \in W, A_2 \in W \Rightarrow (A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2 \qquad (A_1 + A_2 \in W)$

$k \in R, A \in W \Rightarrow (kA)^T = kA^T = kA \qquad (kA \in W)$

$\therefore W \text{ is a subspace of } M_{2\times 2}$

- Ex 3: (The set of singular matrices is not a subspace of $M_{2\times2}$)

  Let W be the set of singular matrices of order 2. Show that W is not a subspace of $M_{2\times2}$ with the standard operations.

Sol:

$$A=\begin{bmatrix}1&0\\0&0\end{bmatrix}\in W,\ B=\begin{bmatrix}0&0\\0&1\end{bmatrix}\in W$$

$$\therefore A+B=\begin{bmatrix}1&0\\0&1\end{bmatrix}\notin W$$

$\therefore W_2$ is not a subspace of $M_{2\times2}$

- Ex 4: (The set of first-quadrant vectors is not a subspace of R^2)

Show that $W=\{(x_1,x_2): x_1 \geq 0 \text{ and } x_2 \geq 0\}$, with the standard operations, is not a subspace of R^2.

Sol:

Let $\mathbf{u}=(1,1) \in W$

$$\because (-1)\mathbf{u}=(-1)(1,1)=(-1,-1) \notin W$$

(not closed under scalar multiplication)

$\therefore W$ is not a subspace of R^2

- Ex 6: (Determining subspaces of R^2)

Which of the following two subsets is a subspace of R^2?

(a) The set of points on the line given by $x+2y=0$.

(b) The set of points on the line given by $x+2y=1$.

Sol:

(a) $W=\{(x,y) \mid x+2y=0\}=\{(-2t,t) \mid t\in R\}$

Let $v_1=(-2t_1,t_1)\in W \quad v_2=(-2t_2,t_2)\in W$

$\because v_1+v_2=(-2(t_1+t_2),t_1+t_2)\in W$ (closed under addition)

$kv_1=(-2(kt_1),kt_1)\in W$ (closed under scalar multiplication)

$\therefore W$ is a subspace of R^2

(b) $W = \{(x, y) \mid x + 2y = 1\}$ (Note: the zero vector is not on the line)

Let $v = (1,0) \in W$

$\because (-1)v = (-1,0) \notin W$

$\therefore W$ is not a subspace of R^2

- Ex 8: (Determining subspaces of R^3)

Which of the following subsets is a subspace of R^3?

(a) $W = \{(x_1, x_2, 1) \mid x_1, x_2 \in R\}$

(b) $W = \{(x_1, x_1 + x_3, x_3) \mid x_1, x_3 \in R\}$

Sol:

(a) Let $\mathbf{v} = (0,0,1) \in W$

$\Rightarrow (-1)\mathbf{v} = (0,0,-1) \notin W$

$\therefore W$ is not a subspace of R^3

(b) Let $\mathbf{v} = (\text{v}_1, \text{v}_1 + \text{v}_3, \text{v}_3) \in W$, $\mathbf{u} = (\text{u}_1, \text{u}_1 + \text{u}_3, \text{u}_3) \in W$

$\because \mathbf{v} + \mathbf{u} = \left(\text{v}_1 + \text{u}_1, (\text{v}_1 + \text{u}_1) + (\text{v}_3 + \text{u}_3), \text{v}_3 + \text{u}_3\right) \in \text{W}$

$k\mathbf{v} = \left(k\text{v}_1, (k\text{v}_1) + (k\text{v}_3), k\text{v}_3\right) \in \text{W}$

$\therefore W$ is a subspace of R^3

- Thm 4.6: (The intersection of two subspaces is a subspace)

If V and W are both subspaces of a vector space U, then the intersection of V and W (denoted by $V \cap U$) is also a subspace of U.

Key Learning in Section 4.3

- Determine whether a subset W of a vector space V is a subspace of V.
- Determine subspaces of R^n.

Keywords in Section 4.3:

- subspace：子空間
- trivial subspace：顯然子空間

4.4 Spanning Sets and Linear Independence

- Linear combination:

A vector $\mathbf{v}$ in a vector space V is called a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ in V if $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_k\mathbf{u}_k \qquad c_1, c_2, \cdots, c_k : \text{scalars}$$

- Ex 2-3: (Finding a linear combination)

$$\mathbf{v}_1=(1,2,3)\quad \mathbf{v}_2=(0,1,2)\quad \mathbf{v}_3=(-1,0,1)$$

Prove (a) $\mathbf{w}=(1,1,1)$ is a linear combination of $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$

(b) $\mathbf{w}=(1,-2,2)$ is not a linear combination of $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$

Sol:

(a) $\mathbf{w}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3$

$$\begin{aligned}(1,1,1)&=c_1(1,2,3)+c_2(0,1,2)+c_3(-1,0,1)\\ &=(c_1-c_3,\ 2c_1+c_2,\ 3c_1+2c_2+c_3)\end{aligned}$$

$$\Rightarrow \begin{aligned} c_1\qquad\quad -c_3 &= 1\\ 2c_1+\ c_2\qquad\qquad &= 1\\ 3c_1+2c_2+c_3 &= 1\end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{\text{Guass–Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow c_1 = 1 + t\,,\; c_2 = -1 - 2t\,,\; c_3 = t$$

(this system has infinitely many solutions)

$$\overset{t=1}{\Rightarrow} \mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$$

(b)

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{\text{Guass–Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

$$\Rightarrow \text{ this system has no solution } (\because 0 \neq 7)$$

$$\Rightarrow \mathbf{w} \neq c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

- the span of a set: span (S)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then **the span of S** is the set of all linear combinations of the vectors in S,

$$\text{span}(S) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \mid \forall c_i \in R\}$$

(the set of all linear combinations of vectors in S)

- a spanning set of a vector space:

If every vector in a given vector space can be written as a linear combination of vectors in a given set S, then S is called **a spanning set** of the vector space.

- Notes:

$$\text{span}(S) = V$$

$$\Rightarrow \; S \text{ spans (generates) } V$$

$$V \text{ is spanned (generated) by } S$$

$$S \text{ is a spanning set of } V$$

- Notes:

(1) $\text{span}(\phi) = \{\mathbf{0}\}$

(2) $S \subseteq \text{span}(S)$

(3) $S_1, S_2 \subseteq V$

$$S_1 \subseteq S_2 \Rightarrow \text{span}(S_1) \subseteq \text{span}(S_2)$$

- Ex 5: (A spanning set for R^3)

Show that the set $S=\{(1,2,3),(0,1,2),(-2,0,1)\}$ sapns R^3

Sol:

We must determine whether an arbitrary vector $\mathbf{u}=(u_1,u_2,u_3)$ in R^3 can be as a linear combination of $\mathbf{v}_1,\mathbf{v}_2,$ and $\mathbf{v}_3$.

$$\mathbf{u}\in R^3 \Rightarrow \mathbf{u}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3$$

$$\begin{aligned} \Rightarrow\ c_1 \qquad\quad -2c_3 &= u_1 \\ 2c_1+\ c_2 \qquad\quad &= u_2 \\ 3c_1+2c_2+c_3 &= u_3 \end{aligned}$$

The problem thus reduces to determining whether this system is consistent for all values of $u_1, u_2,$ and u_3.

$$\because |A| = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} \neq 0$$

$\Rightarrow A\mathbf{x} = \mathbf{b}$ has exactly one solution for every u.

$$\Rightarrow span(S) = R^3$$

- Thm 4.7: (Span(S) is a subspace of V)

If $S=\{\mathbf{v}_1, \mathbf{v}_2,\ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V,

then

(a) span (S) is a subspace of V.

(b) span (S) is the smallest subspace of V that contains S.

(Every other subspace of V that contains S must contain span (S).)

- Linear Independent (L.I.) and Linear Dependent (L.D.):

$S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_k\}$: a set of vectors in a vector space V

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_k\mathbf{v}_k=\mathbf{0}$$

(1) If the equation has only the trivial solution $(c_1=c_2=\cdots=c_k=0)$
then S is called linearly independent.

(2) If the equation has a nontrivial solution (i.e., not all zeros),
then S is called linearly dependent.

- Notes:

(1) ϕ is linearly independent

(2) $\mathbf{0} \in S \Rightarrow S$ is linearly dependent.

(3) $\mathbf{v} \neq \mathbf{0} \Rightarrow \{\mathbf{v}\}$ is linearly independent

(4) $S_1 \subseteq S_2$

S_1 is linearly dependent $\Rightarrow S_2$ is linearly dependent

S_2 is linearly independent $\Rightarrow S_1$ is linearly independent

- Ex 8: (Testing for linearly independent)

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S = \{\underset{\mathbf{v}_1}{(1, 2, 3)}, \underset{\mathbf{v}_2}{(0, 1, 2)}, \underset{\mathbf{v}_3}{(-2, 0, 1)}\}$$

Sol:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \Rightarrow \begin{aligned} c_1 \quad\quad\;\; - 2c_3 &= 0 \\ 2c_1 + \;\; c_2 + \quad\quad\;\; &= 0 \\ 3c_1 + 2c_2 + \;\; c_3 &= 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \xrightarrow{\text{Gauss - Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow c_1 = c_2 = c_3 = 0 \quad (\text{only the trivial solution})$$

$$\Rightarrow S \text{ is linearly independent}$$

- Ex 9: (Testing for linearly independent)

Determine whether the following set of vectors in P_2 is L.I. or L.D.

$$S = \{\underset{\mathbf{v}_1}{1+x-2x^2},\ \underset{\mathbf{v}_2}{2+5x-x^2},\ \underset{\mathbf{v}_3}{x+x^2}\}$$

Sol:

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3 = \mathbf{0}$$

i.e. $c_1(1+x-2x^2)+c_2(2+5x-x^2)+c_3(x+x^2) = 0+0x+0x^2$

$$\Rightarrow \begin{aligned} c_1+2c_2 \qquad &= 0 \\ c_1+5c_2+c_3 &= 0 \\ -2c_1-c_2+c_3 &= 0 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\text{G.J.}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ This system has infinitely many solutions.
(i.e., This system has nontrivial solutions.)

$\Rightarrow$ S is linearly dependent. (Ex: $c_1=2$, $c_2=-1$, $c_3=3$)

- Ex 10: (Testing for linearly independent)

Determine whether the following set of vectors in 2×2 matrix space is L.I. or L.D.

$$S=\left\{\underset{\mathbf{v}_1}{\begin{bmatrix}2 & 1\\0 & 1\end{bmatrix}},\underset{\mathbf{v}_2}{\begin{bmatrix}3 & 0\\2 & 1\end{bmatrix}},\underset{\mathbf{v}_3}{\begin{bmatrix}1 & 0\\2 & 0\end{bmatrix}}\right\}$$

Sol:

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3=\mathbf{0}$$

$$c_1\begin{bmatrix}2 & 1\\0 & 1\end{bmatrix}+c_2\begin{bmatrix}3 & 0\\2 & 1\end{bmatrix}+c_3\begin{bmatrix}1 & 0\\2 & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} 2c_1 + 3c_2 + c_3 &= 0 \\ c_1 \qquad\qquad &= 0 \\ 2c_2 + 2c_3 &= 0 \\ c_1 + c_2 \qquad &= 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\text{Gauss - Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ $c_1 = c_2 = c_3 = 0$ (This system has only the trivial solution.)

$\Rightarrow$ S is linearly independent.

- Thm 4.8: (A property of linearly dependent sets)

  A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, $k \geq 2$, is linearly independent if and only if at least one of the vectors $\boldsymbol{v}_j$ in S can be written as a linear combination of the other vectors in S.

Pf:

$(\Rightarrow)$ $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}$

$\because S$ is linearly dependent

$\Rightarrow c_i \neq 0$ for some i

$$\Rightarrow \mathbf{v}_i = \frac{c_1}{c_\text{i}}\mathbf{v}_1 + \cdots + \frac{c_{i-1}}{c_i}\mathbf{v}_{i-1} + \frac{c_{i+1}}{c_i}\mathbf{v}_{i+1} + \cdots + \frac{c_k}{c_i}\mathbf{v}_k$$

$(\Leftarrow)$

Let $\mathbf{v}_i = d_1\mathbf{v}_1 + \ldots + d_{i\text{-}1}\mathbf{v}_{i\text{-}1} + d_{i+1}\mathbf{v}_{i+1} + \ldots + d_k\mathbf{v}_k$

$\Rightarrow d_1\mathbf{v}_1 + \ldots + d_{i\text{-}1}\mathbf{v}_{i\text{-}1} - \mathbf{v}_i + d_{i+1}\mathbf{v}_{i+1} + \ldots + d_k\mathbf{v}_k = \mathbf{0}$

$\Rightarrow c_1 = d_1, \ldots, c_{i\text{-}1} = d_{i\text{-}1}, c_i = -1, c_{i+1} = d_{i+1}, \ldots, c_k = d_k$ (nontrivial solution)

$\Rightarrow S$ is linearly dependent

- Corollary to Theorem 4.8:

  Two vectors **u** and **v** in a vector space *V* are linearly dependent if and only if one is a scalar multiple of the other.

Key Learning in Section 4.4

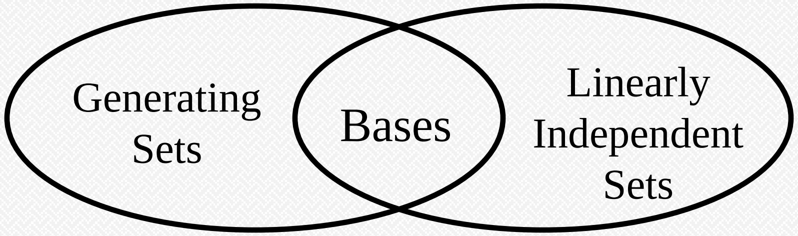
- Write a linear combination of a set of vectors in a vector space *V*.
- Determine whether a set *S* of vectors in a vector space *V* is a spanning set of *V*.
- Determine whether a set of vectors in a vector space *V* is linearly independent.

Keywords in Section 4.4:

- linear combination：線性組合
- spanning set：生成集合
- trivial solution：顯然解
- linear independent：線性獨立
- linear dependent：線性相依

4.5 Basis and Dimension

- **Basis:**



V : a vector space

$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subseteq V$

$$\begin{cases} (a) & S \text{ spans } V \ \text{ (i.e., } span(S) = V\text{)} \\ (b) & S \text{ is linearly independent} \end{cases}$$

$\Rightarrow$ S is called a **basis** for V

- **Notes:**

(1) Ø is a basis for $\{\mathbf{0}\}$

(2) the standard basis for R^3:

$\{i, j, k\} \quad i = (1, 0, 0),\ j = (0, 1, 0),\ k = (0, 0, 1)$

(3) the standard basis for R^n:

$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ $\mathbf{e}_1 = (1,0,\ldots,0)$, $\mathbf{e}_2 = (0,1,\ldots,0)$, $\mathbf{e}_n = (0,0,\ldots,1)$

Ex: R^4 $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

(4) the standard basis for $m \times n$ matrix space:

$\{ E_{ij} \mid 1 \le i \le m , 1 \le j \le n \}$

Ex: 2×2 matrix space:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

(5) the standard basis for $P_n(x)$:

$\{1, x, x^2, \ldots, x^n\}$

Ex: $P_3(x)$ $\{1, x, x^2, x^3\}$

- Thm 4.9: (Uniqueness of basis representation)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of vectors in S.

Pf:

$$\because S \text{ is a basis} \Rightarrow \begin{cases} \textit{1.} \quad span(S) = V \\ \textit{2.} \quad S \text{ is linearly independent} \end{cases}$$

$\because span(S) = V$ Let $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$

$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \ldots + b_n\mathbf{v}_n$

$$\Rightarrow \mathbf{0} = (c_1 - b_1)\mathbf{v}_1 + (c_2 - b_2)\mathbf{v}_2 + \ldots + (c_n - b_n)\mathbf{v}_n$$

$\because S$ is linearly independent

$\Rightarrow c_1 = b_1,\ c_2 = b_2, \ldots, c_n = b_n$ (i.e., uniqueness)

- Thm 4.10: (Bases and linear dependence)

  If $S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ is a basis for a vector space V, then every set containing more than n vectors in V is linearly dependent.

Pf:

Let $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$, $m > n$

$\because \text{span}(S)=V$

$$\mathbf{u}_i \in V \quad \Rightarrow \quad \begin{aligned} \mathbf{u}_1 &= c_{11}\mathbf{v}_1 + c_{21}\mathbf{v}_2 + \cdots + c_{n1}\mathbf{v}_n \\ \mathbf{u}_2 &= c_{12}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + \cdots + c_{n2}\mathbf{v}_n \\ &\vdots \\ \mathbf{u}_m &= c_{1m}\mathbf{v}_1 + c_{2m}\mathbf{v}_2 + \cdots + c_{nm}\mathbf{v}_n \end{aligned}$$

Let $k_1\mathbf{u}_1+k_2\mathbf{u}_2+\ldots+k_m\mathbf{u}_m=\mathbf{0}$

$\Rightarrow d_1\mathbf{v}_1+d_2\mathbf{v}_2+\ldots+d_n\mathbf{v}_n=\mathbf{0}$ (where $d_i=c_{i1}k_1+c_{i2}k_2+\ldots+c_{im}k_m$)

$\because S$ is L.I.

$\Rightarrow d_i=0 \quad \forall i$ i.e.

$$\begin{gathered} c_{11}k_1+c_{12}k_2+\cdots+c_{1m}k_m=0 \\ c_{21}k_1+c_{22}k_2+\cdots+c_{2m}k_m=0 \\ \vdots \\ c_{n1}k_1+c_{n2}k_2+\cdots+c_{nm}k_m=0 \end{gathered}$$

$\because$ Thm 1.1: If the homogeneous system has fewer equations than variables, then it must have infinitely many solution.

$m>n \Rightarrow k_1\mathbf{u}_1+k_2\mathbf{u}_2+\ldots+k_m\mathbf{u}_m=\mathbf{0}$ has nontrivial solution

$\Rightarrow S_1$ is linearly dependent

- Thm 4.11: (Number of vectors in a basis)

  If a vector space V has one basis with n vectors, then every basis for V has n vectors. (All bases for a finite-dimensional vector space has the same number of vectors.)

Pf:

$S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$
$S'=\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ two bases for a vector space

$$\left.\begin{array}{l}\left.\begin{array}{r} S \text{ is a basis} \\ S' \text{ is L.I.}\end{array}\right\} \overset{Thm.4.10}{\Rightarrow} n \geq m \\ \left.\begin{array}{r} S \text{ is L.I.} \\ S' \text{ is a basis}\end{array}\right\} \overset{Thm.4.10}{\Rightarrow} n \leq m \end{array}\right\} \Rightarrow n = m$$

- Finite dimensional:

  A vector space V is called **finite dimensional**,

  if it has a basis consisting of a finite number of elements.

- Infinite dimensional:

  If a vector space V is not finite dimensional,

  then it is called **infinite dimensional**.

- Dimension:

  The **dimension** of a finite dimensional vector space V is

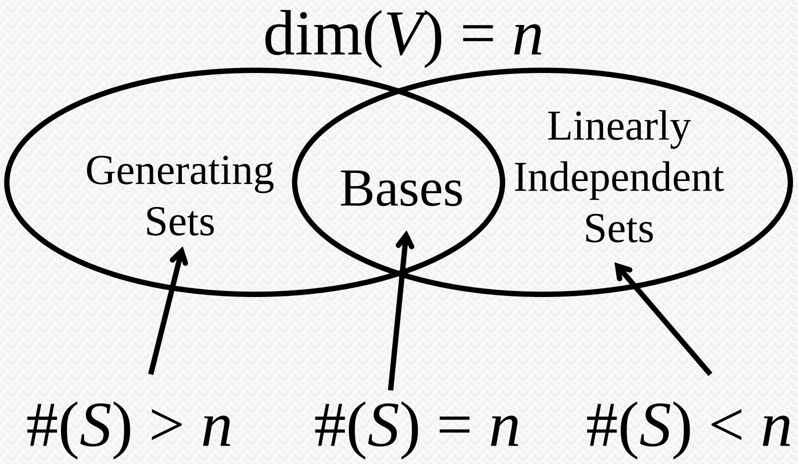
  defined to be the number of vectors in a basis for V.

  V: a vector space $\qquad$ S: a basis for V

  $\Rightarrow \dim(V) = \#(S)$ $\qquad$ (the number of vectors in S)

- Notes:

(1) $\dim(\{\mathbf{0}\}) = 0 = \#(\varnothing)$



(2) $\dim(V) = n$, $S \subseteq V$

S : a generating set $\Rightarrow \#(S) \geq n$

S : a L.I. set $\Rightarrow \#(S) \leq n$

S : a basis $\Rightarrow \#(S) = n$

(3) $\dim(V) = n$, W is a subspace of V $\Rightarrow \dim(W) \leq n$

- Ex:

(1) Vector space R^n $\Rightarrow$ basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$

$\Rightarrow \dim(R^n) = n$

(2) Vector space $M_{m\times n}$ $\Rightarrow$ basis $\{E_{ij} \mid 1\le i\le m, 1\le j\le n\}$

$\Rightarrow \dim(M_{m\times n}) = mn$

(3) Vector space $P_n(x)$ $\Rightarrow$ basis $\{1, x, x^2, \ldots, x^n\}$

$\Rightarrow \dim(P_n(x)) = n+1$

(4) Vector space $P(x)$ $\Rightarrow$ basis $\{1, x, x^2, \ldots\}$

$\Rightarrow \dim(P(x)) = \infty$

- Ex 9: (Finding the dimension of a subspace)

(a) $W=\{(d, c-d, c)$: c and d are real numbers$\}$

(b) $W=\{(2b, b, 0)$: b is a real number$\}$

Sol: (Note: Find a set of L.I. vectors that spans the subspace)

(a) $(d, c-d, c) = c(0, 1, 1) + d(1, -1, 0)$

$\Rightarrow S = \{(0, 1, 1), (1, -1, 0)\}$ (S is L.I. and S *spans* W)

$\Rightarrow S$ is a basis for W

$\Rightarrow \dim(W) = \#(S) = 2$

(b) $\because (2b, b, 0) = b(2, 1, 0)$

$\Rightarrow S = \{(2, 1, 0)\}$ spans W and S is L.I.

$\Rightarrow S$ is a basis for W

$\Rightarrow \dim(W) = \#(S) = 1$

- Ex 11: (Finding the dimension of a subspace)

  Let W be the subspace of all symmetric matrices in $M_{2\times 2}$.

  What is the dimension of W?

Sol:

$$W = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| a, b, c \in R \right\}$$

$$\because \begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ spans } W \text{ and } S \text{ is L.I.}$$

$\Rightarrow S$ is a basis for W $\quad \Rightarrow \dim(W) = \#(S) = 3$

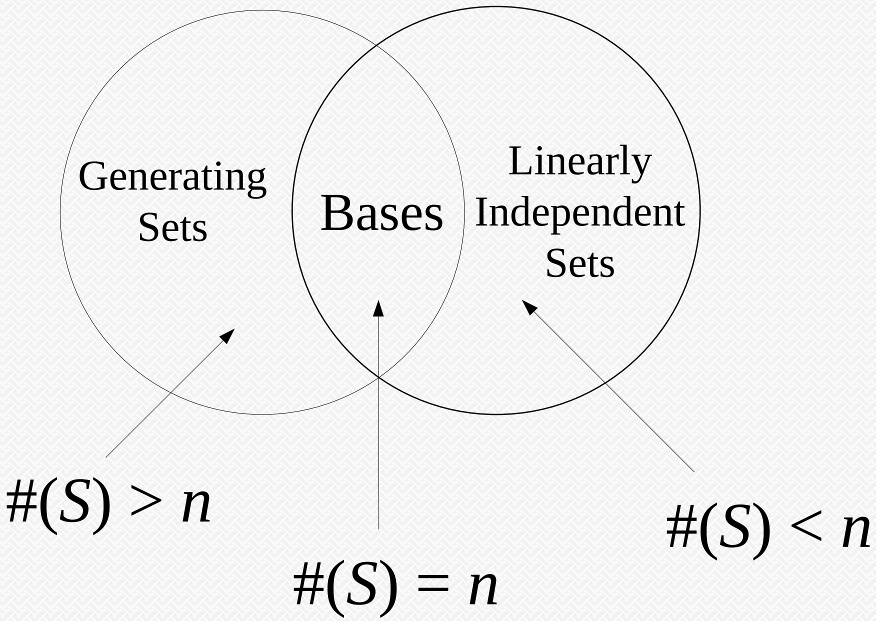
- Thm 4.12: (Basis tests in an n-dimensional space)

Let V be a vector space of dimension n.

(1) If $S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ is a linearly independent set of vectors in V, then S is a basis for V.

(2) If $S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ spans V, then S is a basis for V.

$\dim(V)=n$



Key Learning in Section 4.5

- Recognize bases in the vector spaces R^n, P_n and $M_{m,n}$
- Find the dimension of a vector space.

Keywords in Section 4.5

- basis：基底
- dimension：維度
- finite dimension：有限維度
- infinite dimension：無限維度

4.6 Rank of a Matrix and Systems of Linear Equations

- row vectors:

Row vectors of A

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}=\begin{bmatrix} A_{(1)} \\ A_{(2)} \\ \vdots \\ A_{(m)} \end{bmatrix}$$

$$[a_{11}, a_{12}, \ldots, a_{1n}] = A_{(1)}$$
$$[a_{21}, a_{22}, \ldots, a_{2n}] = A_{(2)}$$
$$\vdots$$
$$[a_{m1}, a_{m2}, \ldots, a_{mn}] = A_{(n)}$$

- column vectors:

Column vectors of A

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}=\left[A^{(1)} \vdots A^{(2)} \vdots \cdots \vdots A^{(n)}\right]$$

$$\underset{\displaystyle A^{(1)}}{\underset{\|}{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}}} \underset{\displaystyle A^{(2)}}{\underset{\|}{\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}}} \cdots \underset{\displaystyle A^{(n)}}{\underset{\|}{\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}}}$$

Let A be an $m \times n$ matrix.

- Row space:

  The **row space** of A is the subspace of R^n spanned by the row vectors of A.

$$RS(A) = \{\alpha_1 A_{(1)} + \alpha_2 A_{(2)} + \ldots + \alpha_m A_{(m)} \mid \alpha_1, \alpha_2, \ldots, \alpha_m \in R\}$$

- Column space:

  The **column space** of A is the subspace of R^m spanned by the column vectors of A.

$$CS(A) = \{\beta_1 A^{(1)} + \beta_2 A^{(2)} + \cdots + \beta_n A^{(n)} \mid \beta_1, \beta_2, \cdots \beta_n \in R\}$$

- Null space:

  The **null space** of A is the set of all solutions of $A\mathbf{x}=\mathbf{0}$ and it is a subspace of R^n.

$$NS(A) = \{\mathbf{x} \in R^n \mid A\mathbf{x} = \mathbf{0}\}$$

- Thm 4.13: (Row-equivalent matrices have the same row space)

  If an $m \times n$ matrix A is row equivalent to an $m \times n$ matrix B,

  then the row space of A is equal to the row space of B.

- Notes:

  (1) The row space of a matrix is not changed by elementary row operations.

  $RS(r(A)) = RS(A)$ $\qquad$ r: elementary row operations

  (2) Elementary row operations can change the column space.

- Thm 4.14: (Basis for the row space of a matrix)

  If a matrix A is row equivalent to a matrix B in row-echelon form, then the nonzero row vectors of B form a basis for the row space of A.

- Ex 2: (Finding a basis for a row space)

Find a basis of row space of $A=\begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & 2 \end{bmatrix}$

Sol:

$$A=\begin{array}{c} \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & 2 \end{bmatrix} \\ \begin{matrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{matrix} \end{array} \xrightarrow{\text{G.E.}} B=\begin{array}{cc} \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \\ \\ \end{matrix} \\ \begin{matrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{matrix} & \end{array}$$

a basis for $RS(A)$ = {the nonzero row vectors of B} (Thm 4.14)

= $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ = {(1, 3, 1, 3), (0, 1, 1, 0), (0, 0, 0, 1)}

- Notes:

(1) $\mathbf{b}_3 = -2\mathbf{b}_1 + \mathbf{b}_2 \Rightarrow \mathbf{a}_3 = -2\mathbf{a}_1 + \mathbf{a}_2$

(2) $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4\}$ is L.I. $\Rightarrow \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is L.I.

- Ex 3: (Finding a basis for a subspace)

Find a basis for the subspace of R^3 spanned by

$$S = \{(\overset{\mathbf{v}_1}{-1, 2, 5}), (\overset{\mathbf{v}_2}{3, 0, 3}), (\overset{\mathbf{v}_3}{5, 1, 8})\}$$

Sol:

$$A = \begin{array}{c} \begin{bmatrix} -1 & 2 & 5 \\ 3 & 0 & 3 \\ 5 & 1 & 8 \end{bmatrix} \end{array} \begin{array}{c} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{array} \xrightarrow{\text{G.E.}} B = \begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{c} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \end{array}$$

a basis for $span(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$

= a basis for $RS(A)$

= {the nonzero row vectors of B} (Thm 4.14)

= $\{\mathbf{w}_1, \mathbf{w}_2\}$

= $\{(1, -2, -5), (0, 1, 3)\}$

- Ex 4-5: (Finding a basis for the column space of a matrix)

Find a basis for the column space of the matrix A given in Ex 2.

$$A=\begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$$

Sol. (Method 1):

$$A^T=\begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 3 & 1 & 0 & 4 & 0 \\ 1 & 1 & 6 & -2 & -4 \\ 3 & 0 & -1 & 1 & -2 \end{bmatrix} \xrightarrow{G.E.} B=\begin{array}{c} \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}\begin{array}{l} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \\ \boldsymbol{w}_3 \\ \\ \end{array}$$

$\because \quad CS(A)=RS(A^T)$

$\therefore$ a basis for $CS(A)$

= a basis for $RS(A^T)$

= {the nonzero vectors of B}

= $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$

$$= \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 9 \\ -5 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}$$ (a basis for the column space of A)

- Note: This basis is not a subset of $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4\}$.

- Sol. (Method 2):

$$A=\begin{array}{c}\begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix} \\ \begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{c}_4 \end{array}\end{array} \xrightarrow{G.E.} B=\begin{array}{c}\begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{array}\end{array}$$

Leading 1 => $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a basis for $CS(B)$

$\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_4\}$ is a basis for $CS(A)$

- Notes:

(1) This basis is a subset of $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4\}$.

(2) $\mathbf{v}_3 = -2\mathbf{v}_1 + \mathbf{v}_2$, thus $\mathbf{c}_3 = -2\mathbf{c}_1 + \mathbf{c}_2$.

- Thm 4.16: (Solutions of a homogeneous system)

  If A is an $m\times n$ matrix, then the set of all solutions of the homogeneous system of linear equations $A\mathbf{x} = \mathbf{0}$ is a subspace of R^n called the nullspace of A.

Pf:

$$NS(A)=\{x\in R^n \mid Ax=0\}$$

$$NS(A)\in R^n$$

$$NS(A)\neq \phi \quad (\because A\mathbf{0}=\mathbf{0})$$

Let $\mathbf{x}_1, \mathbf{x}_2 \in NS(A)$ (i.e. $A\mathbf{x}_1=\mathbf{0}, A\mathbf{x}_2=\mathbf{0}$)

Then (1) $A(\mathbf{x}_1+\mathbf{x}_2)=A\mathbf{x}_1+A\mathbf{x}_2=\mathbf{0}+\mathbf{0}=\mathbf{0}$ Addition

(2) $A(c\mathbf{x}_1)=c(A\mathbf{x}_1)=c(\mathbf{0})=\mathbf{0}$ Scalar multiplication

Thus $NS(A)$ is a subspace of R^n

- Notes: The nullspace of A is also called the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

- Ex 7: (Finding the solution space of a homogeneous system)

Find the nullspace of the matrix *A*.

$$A=\begin{bmatrix}1 & 2 & -2 & 1\\ 3 & 6 & -5 & 4\\ 1 & 2 & 0 & 3\end{bmatrix}$$

Sol: The nullspace of *A* is the solution space of $A\mathbf{x} = \mathbf{0}$.

$$A=\begin{bmatrix}1 & 2 & -2 & 1\\ 3 & 6 & -5 & 4\\ 1 & 2 & 0 & 3\end{bmatrix}\xrightarrow{G.J.E}\begin{bmatrix}1 & 2 & 0 & 3\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$\Rightarrow x_1 = -2s - 3t,\ x_2 = s,\ x_3 = -t,\ x_4 = t$$

$$\Rightarrow \mathbf{x}=\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=\begin{bmatrix}-2s-3t\\ s\\ -t\\ t\end{bmatrix}=s\begin{bmatrix}-2\\ 1\\ 0\\ 0\end{bmatrix}+t\begin{bmatrix}-3\\ 0\\ -1\\ 1\end{bmatrix}=s\mathbf{v}_1+t\mathbf{v}_2$$

$$\Rightarrow NS(A)=\{s\mathbf{v}_1+t\mathbf{v}_2 \mid s,t\in R\}$$

- Thm 4.15: (Row and column space have equal dimensions)

If A is an $m \times n$ matrix, then the row space and the column space of A have the same dimension.

$$\dim(RS(A)) = \dim(CS(A))$$

- Rank:

The dimension of the row (or column) space of a matrix A is called the **rank** of A and is denoted by rank(A).

$$\text{rank}(A) = \dim(RS(A)) = \dim(CS(A))$$

- Nullity:

The dimension of the nullspace of A is called the nullity of A.

$$\text{nullity}(A) = \dim(NS(A))$$

- Note: $\text{rank}(A^T) = \text{rank}(A)$

Pf: $\text{rank}(A^T) = \dim(RS(A^T)) = \dim(CS(A)) = \text{rank}(A)$

- Thm 4.17: (Dimension of the solution space)

If A is an $m \times n$ matrix of rank r, then the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is $n - r$. That is

$$n = \text{rank}(A) + \text{nullity}(A)$$

- Notes:

(1) rank(A): The number of <u>leading variables</u> in the solution of $A\mathbf{x}=\mathbf{0}$.

(The number of nonzero rows in the row-echelon form of A)

(2) nullity (A): The number of <u>free variables</u> in the solution of $A\mathbf{x} = \mathbf{0}$.

- Notes:

If A is an $m \times n$ matrix and rank(A) = r, then

Fundamental Space	Dimension
$RS(A)=CS(A^T)$	r
$CS(A)=RS(A^T)$	r
$NS(A)$	$n-r$
$NS(A^T)$	$m-r$

- Ex 8: (Rank and nullity of a matrix)

Let the column vectors of the matrix A be denoted by $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$, and $\mathbf{a}_5$.

$$A=\begin{array}{c} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix} \\ \begin{array}{ccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{array} \end{array}$$

(a) Find the rank and nullity of A.

(b) Find a subset of the column vectors of A that forms a basis for the column space of A.

(c) If possible, write the third column of A as a linear combination of the first two columns.

Sol: Let B be the reduced row-echelon form of A.

$$A=\begin{array}{c} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix} \\ \begin{matrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{matrix} \end{array} \qquad B=\begin{array}{c} \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5 \end{matrix} \end{array}$$

(a) $\text{rank}(A) = 3$ (the number of nonzero rows in B)

$$\text{nuillity }(A)=n-\text{rank}(A)=5-3=2$$

(b) Leading 1

$\Rightarrow \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4\}$ is a basis for $CS(B)$

$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is a basis for $CS(A)$

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \text{ and } \mathbf{a}_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix},$$

(c) $\mathbf{b}_3 = -2\mathbf{b}_1 + 3\mathbf{b}_2 \Rightarrow \ \mathbf{a}_3 = -2\mathbf{a}_1 + 3\mathbf{a}_2$

- Thm 4.18: (Solutions of a nonhomogeneous linear system)

If $\mathbf{x}_p$ is a particular solution of the nonhomogeneous system $A\mathbf{x} = \boldsymbol{b}$, then every solution of this system can be written in the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, wher $\mathbf{x}_h$ is a solution of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$.

Pf:

Let $\mathbf{x}$ be any solution of $A\mathbf{x} = \boldsymbol{b}$.

$$\Rightarrow A(\mathbf{x}-\mathbf{x}_p) = A\mathbf{x} - A\mathbf{x}_p = \mathbf{b}-\mathbf{b} = \mathbf{0}.$$

$\Rightarrow (\mathbf{x}-\mathbf{x}_p)$ is a solution of $A\mathbf{x} = \mathbf{0}$

Let $\mathbf{x}_h = \mathbf{x}-\mathbf{x}_p$

$$\Rightarrow \mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$$

- Ex 9: (Finding the solution set of a nonhomogeneous system)

Find the set of all solution vectors of the system of linear equations.

$$\begin{array}{rcrcrcrcr} x_1 & & & - & 2x_3 & + & x_4 & = & 5 \\ 3x_1 & + & x_2 & - & 5x_3 & & & = & 8 \\ x_1 & + & 2x_2 & & & - & 5x_4 & = & -9 \end{array}$$

Sol:

$$\left[\begin{array}{rrrr:r} 1 & 0 & -2 & 1 & 5 \\ 3 & 1 & -5 & 0 & 8 \\ 1 & 2 & 0 & -5 & -9 \end{array}\right] \xrightarrow{G.J.E} \begin{array}{c} \left[\begin{array}{rrrr:r} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{rrrrr} & & s & t & \end{array} \end{array}$$

$$\Rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s & - & t & + & 5 \\ -s & + & 3t & - & 7 \\ s & + & 0t & + & 0 \\ 0s & + & t & + & 0 \end{bmatrix} = s\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix}$$

$$= s\mathbf{u}_1 + t\mathbf{u}_2 + \mathbf{x}_p$$

i.e. $\mathbf{x}_p = \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution vector of $A\mathbf{x}=\mathbf{b}$.

$\mathbf{x}_h = s\mathbf{u}_1 + t\mathbf{u}_2$ is a solution of $A\mathbf{x} = \mathbf{0}$

- Thm 4.19: (Solution of a system of linear equations)

The system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

Pf:

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

be the coefficient matrix, the column matrix of unknowns, and the right-hand side, respectively, of the system $A\mathbf{x} = \mathbf{b}$.

Then

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n \\ \vdots & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

Hence, $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is a linear combination of the columns of A. That is, the system is consistent if and only if $\mathbf{b}$ is in the subspace of R^m spanned by the columns of A.

- Note:

If rank([A|**b**])=rank(A)

Then the system $A\mathbf{x}=\mathbf{b}$ is consistent.

- Ex 10: (Consistency of a system of linear equations)

$$
\begin{array}{rcrcrcr}
x_1 & + & x_2 & - & x_3 & = & -1 \\
x_1 & & & + & x_3 & = & 3 \\
3x_1 & + & 2x_2 & - & x_3 & = & 1
\end{array}
$$

Sol:

$$
A=\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 3 & 2 & -1 \end{bmatrix} \xrightarrow{G.J.E.} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$[A \vdots \mathbf{b}] = \begin{array}{c} \left[\begin{array}{ccc:c} 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 3 \\ 3 & 2 & -1 & 1 \end{array}\right] \\ \begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{b} \end{array} \end{array} \xrightarrow{G.J.E.} \begin{array}{c} \left[\begin{array}{ccc:c} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \mathbf{v} \end{array} \end{array}$$

$$\because \mathbf{v} = 3\mathbf{w}_1 - 4\mathbf{w}_2$$

$$\Rightarrow \mathbf{b} = 3\mathbf{c}_1 - 4\mathbf{c}_2 + 0\mathbf{c}_3 \quad (\mathbf{b} \text{ is in the column space of } A)$$

$\Rightarrow$ The system of linear equations is consistent.

- Check:

$$\text{rank}(A) = \text{rank}([A \vdots \mathbf{b}]) = 2$$

- Summary of equivalent conditions for square matrices:

If A is an $n\times n$ matrix, then the following conditions are equivalent.

(1) A is invertible

(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $n\times 1$ matrix $\mathbf{b}$.

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

(4) A is row-equivalent to I_n

(5) $|A| \neq 0$

(6) $\text{rank}(A) = n$

(7) The n row vectors of A are linearly independent.

(8) The n column vectors of A are linearly independent.

Key Learning in Section 4.6

- Find a basis for the row space, a basis for the column space, and the rank of a matrix.
- Find the nullspace of a matrix.
- Find the solution of a consistent system $A\mathbf{x} = \mathbf{b}$ in the form $\mathbf{x}_p + \mathbf{x}_h$.

Keywords in Section 4.6:

- row space : 列空間
- column space : 行空間
- null space: 零空間
- solution space : 解空間
- rank: 秩
- nullity : 核次數

4.7 Coordinates and Change of Basis

- Coordinate representation relative to a basis

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an ordered basis for a vector space V and let $\mathbf{x}$ be a vector in V such that

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

The scalars $c_1, c_2, \ldots, c_n$ are called the **coordinates of x relative to the basis *B***. The **coordinate matrix** (or **coordinate vector**) of $\mathbf{x}$ relative to B is the column matrix in R^n whose components are the coordinates of $\mathbf{x}$.

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

- Ex 1: (Coordinates and components in R^n)

Find the coordinate matrix of $\mathbf{x} = (-2, 1, 3)$ in R^3

relative to the standard basis

$$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

Sol:

$$\because \mathbf{x} = (-2, 1, 3) = -2(1, 0, 0) + 1(0, 1, 0) + 3(0, 0, 1),$$

$$\therefore [\mathbf{x}]_S = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}.$$

- Ex 3: (Finding a coordinate matrix relative to a nonstandard basis)

Find the coordinate matrix of $\mathbf{x} = (1, 2, -1)$ in R^3

relative to the (nonstandard) basis

$$B' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}=\{(1, 0, 1), (0, -1, 2), (2, 3, -5)\}$$

Sol: $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 \Rightarrow (1, 2, -1) = c_1(1, 0, 1) + c_2(0, -1, 2) + c_3(2, 3, -5)$

$$\Rightarrow \begin{array}{rcrcrcr} c_1 & & & + & 2c_3 & = & 1 \\ & & -c_2 & + & 3c_3 & = & 2 \\ c_1 & + & 2c_2 & - & 5c_3 & = & -1 \end{array} \quad \text{i.e.} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 3 & 2 \\ 1 & 2 & -5 & -1 \end{bmatrix} \xrightarrow{\text{G.J.E.}} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & -2 \end{bmatrix} \Rightarrow [\boldsymbol{x}]_{B'} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}$$

- Change of basis problem:

  You were given the coordinates of a vector relative to one basis B and were asked to find the coordinates relative to another basis B'.

- Ex: (Change of basis)

  Consider two bases for a vector space V

$$B=\{\mathbf{u}_1,\mathbf{u}_2\},\ B'=\{\mathbf{u}_1',\mathbf{u}_2'\}$$

$$\text{If } [\mathbf{u}_1']_B=\begin{bmatrix}a\\ b\end{bmatrix},\ [\mathbf{u}_2']_B=\begin{bmatrix}c\\ d\end{bmatrix}$$

$$\text{i.e., } \mathbf{u}_1'=a\mathbf{u}_1+b\mathbf{u}_2,\quad \mathbf{u}_2'=c\mathbf{u}_1+d\mathbf{u}_2$$

Let $\mathbf{v} \in V$, $[\mathbf{v}]_{B'} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$

$$\begin{aligned}
\Rightarrow \mathbf{v} &= k_1\mathbf{u}'_1 + k_2\mathbf{u}'_2 \\
&= k_1(a\mathbf{u}_1 + b\mathbf{u}_2) + k_2(c\mathbf{u}_1 + d\mathbf{u}_2) \\
&= (k_1a + k_2c)\mathbf{u}_1 + (k_1b + k_2d)\mathbf{u}_2
\end{aligned}$$

$$\begin{aligned}
\Rightarrow [\mathbf{v}]_B &= \begin{bmatrix} k_1a + k_2c \\ k_1b + k_2d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \\
&= \left[[\mathbf{u}'_1]_B \; [\mathbf{u}'_2]_B\right] [\mathbf{v}]_{B'}
\end{aligned}$$

- Transition matrix from *B*' to *B*:

Let $B=\{\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_n\}$ and $B'=\{\mathbf{u}'_1,\mathbf{u}'_2...,\mathbf{u}'_n\}$ be two bases for a vector space V

If $[\mathbf{v}]_B$ is the coordinate matrix of **v** relative to *B*

$[\mathbf{v}]_{B'}$ is the coordinate matrix of **v** relative to *B*'

then $[\mathbf{v}]_B=P[\mathbf{v}]_{B'}$

$$=\left[[\mathbf{u}'_1]_B,[\mathbf{u}'_2]_B,...,[\mathbf{u}'_n]_B\right][v]_{B'}$$

where

$$P=\left[[\mathbf{u}'_1]_B,[\mathbf{u}'_2]_B,...,[\mathbf{u}'_n]_B\right]$$

is called **the transition matrix from *B*' to *B***

- Thm 4.20: (The inverse of a transition matrix)

  If P is the transition matrix from a basis B' to a basis B in R^n, then

  (1) P is invertible

  (2) The transition matrix from B to B' is P^{-1}

- Notes:

$$B=\{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n\},\qquad B'=\{\mathbf{u}'_1,\mathbf{u}'_2,\ldots,\mathbf{u}'_n\}$$

$$[\mathbf{v}]_B=\big[[\mathbf{u}'_1]_B,[\mathbf{u}'_2]_B,\ldots,[\mathbf{u}'_n]_B\big]\,[\mathbf{v}]_{B'}=P\,[\mathbf{v}]_{B'}$$

$$[\mathbf{v}]_{B'}=\big[[\mathbf{u}_1]_{B'},[\mathbf{u}_2]_{B'},\ldots,[\mathbf{u}_n]_{B'}\big]\,[\mathbf{v}]_B=P^{-1}\,[\mathbf{v}]_B$$

- Thm 4.21: (Transition matrix from B to B')

Let $B=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $B'=\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be two bases for R^n. Then the transition matrix P^{-1} from B to B' can be found by using Gauss-Jordan elimination on the $n\times 2n$ matrix $[B' \vdots B]$ as follows.

$$[B' \vdots B] \longrightarrow [I_n \vdots P^{-1}]$$

- Ex 5: (Finding a transition matrix)

$B=\{(-3, 2), (4,-2)\}$ and $B'=\{(-1, 2), (2,-2)\}$ are two bases for R^2

(a) Find the transition matrix from B' to B.

(b) Let $[\mathbf{v}]_{B'}=\begin{bmatrix}1\\2\end{bmatrix}$, find $[\mathbf{v}]_B$

(c) Find the transition matrix from B to B' .

Sol:

(a) $$\left[\begin{array}{cc:cc} -3 & 4 & -1 & 2 \\ 2 & -2 & 2 & -2 \end{array}\right] \xrightarrow{\text{G.J.E.}} \left[\begin{array}{cc:cc} 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$$\quad B \qquad\qquad B' \qquad\qquad\qquad I \qquad P$$

$$\therefore P = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$$ (the transition matrix from B' to B)

(b)

$$[\boldsymbol{v}]_{B'} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow [\boldsymbol{v}]_{B} = P[\boldsymbol{v}]_{B'} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Check :

$$[\boldsymbol{v}]_{B'} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \boldsymbol{v} = (1)(-1,2) + (2)(2,-2) = (3,-2)$$

$$[\boldsymbol{v}]_{B} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \Rightarrow \boldsymbol{v} = (-1)(3,-2) + (0)(4,-2) = (3,-2)$$

(c)

$$\left[\begin{array}{cc:cc} -1 & 2 & -3 & 4 \\ 2 & -2 & 2 & -2 \end{array}\right] \xrightarrow{\text{G.J.E.}} \left[\begin{array}{cc:cc} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 3 \end{array}\right]$$

$$B' \qquad\qquad B \qquad\qquad\qquad\qquad I \qquad\qquad P^{-1}$$

$$\therefore P^{-1} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \quad (\text{the transition matrix from } B \text{ to } B')$$

- Check:

$$PP^{-1} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

- Ex 6: (Coordinate representation in $P_3(x)$)

(a) Find the coordinate matrix of $p = 3x^3 - 2x^2 + 4$ relative to the standard basis $S = \{1, x, x^2, x^3\}$ in $P_3(x)$.

(b) Find the coordinate matrix of $p = 3x^3 - 2x^2 + 4$ relative to the basis $S = \{1, 1+x, 1+x^2, 1+x^3\}$ in $P_3(x)$.

Sol:

$$\text{(a) } p = (4)(1) + (0)(x) + (\text{-}2)(x^2) + (3)(x^3) \Rightarrow [p]_B = \begin{bmatrix} 4 \\ 0 \\ -2 \\ 3 \end{bmatrix}$$

$$\text{(b) } p = (3)(1) + (0)(1+x) + (\text{-}2)(1+x^2) + (3)(1+x^3) \Rightarrow [p]_B = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 3 \end{bmatrix}$$

- Ex: (Coordinate representation in M_{2x2})

Find the coordinate matrix of $x = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ relative to the standard basis in $M_{2\times2}$.

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Sol:

$$x = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = 5\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 6\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 7\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 8\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow [x]_B = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$$

Key Learning in Section 4.7

- Find a coordinate matrix relative to a basis in R^n
- Find the transition matrix from the basis to the basis B' in R^n.
- Represent coordinates in general n-dimensional spaces.

Keywords in Section 4.7

- coordinates of **x** relative to B: **x**相對於B的座標
- coordinate matrix: 座標矩陣
- coordinate vector: 座標向量
- change of basis problem: 基底變換問題
- transition matrix from B' to B: 從 B' 到 B的轉移矩陣

4.8 Applications of Vector Spaces

- Conic sections and rotation:

Every conic section in the xy-plane has an equation that can be written in the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

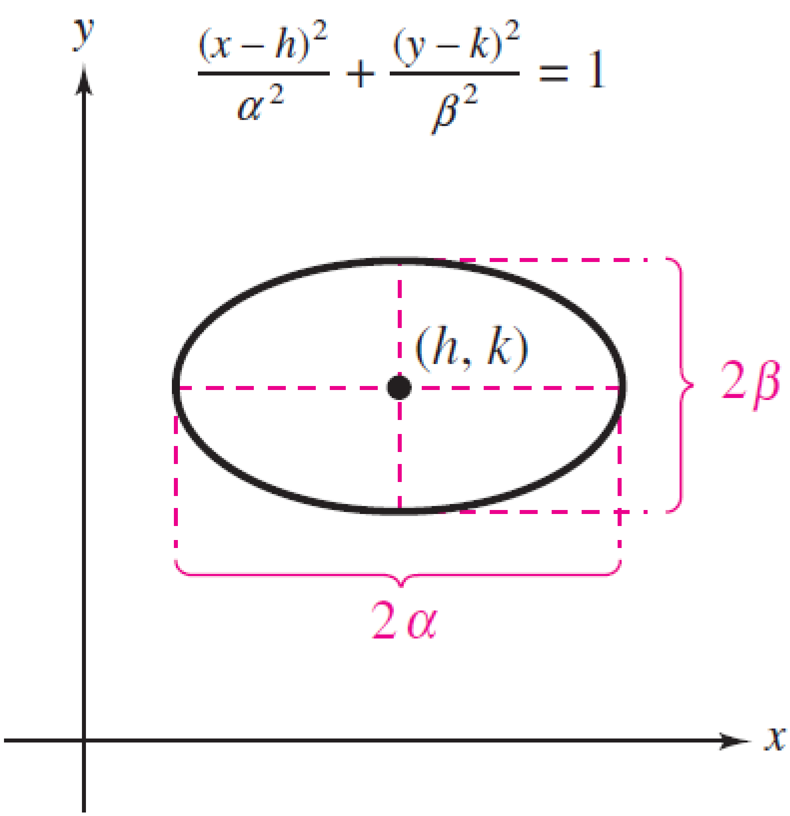
Identifying the graph of this equation is fairly simple as long as b, the coefficient of the xy-term, is zero.

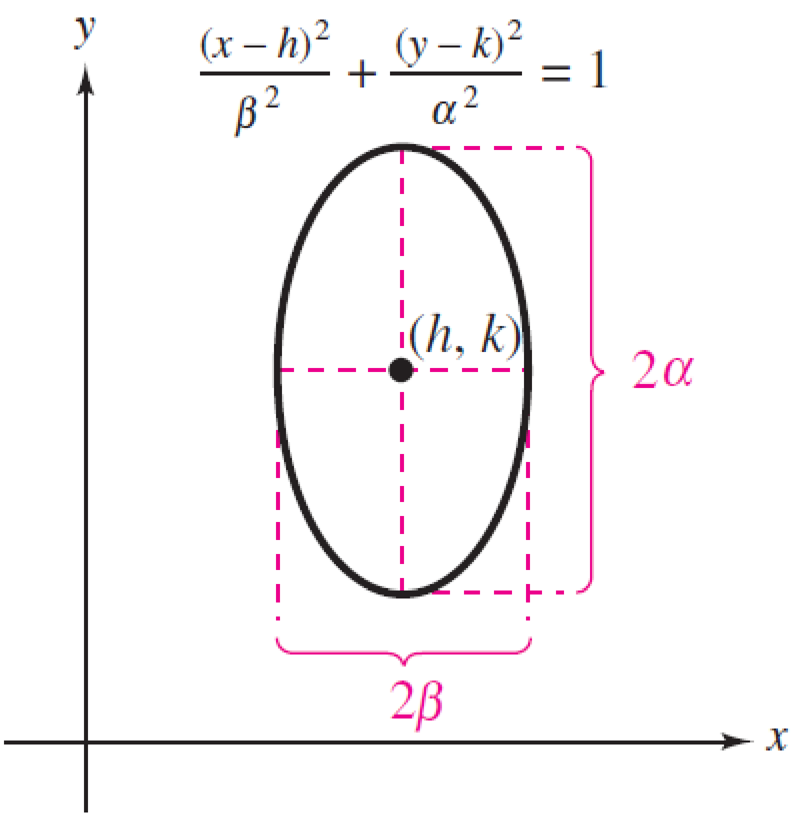
When b is zero, the conic axes are parallel to the coordinate axes, and the identification is accomplished by writing the equation in standard (completed square) form.

- Standard forms of equations of conics:

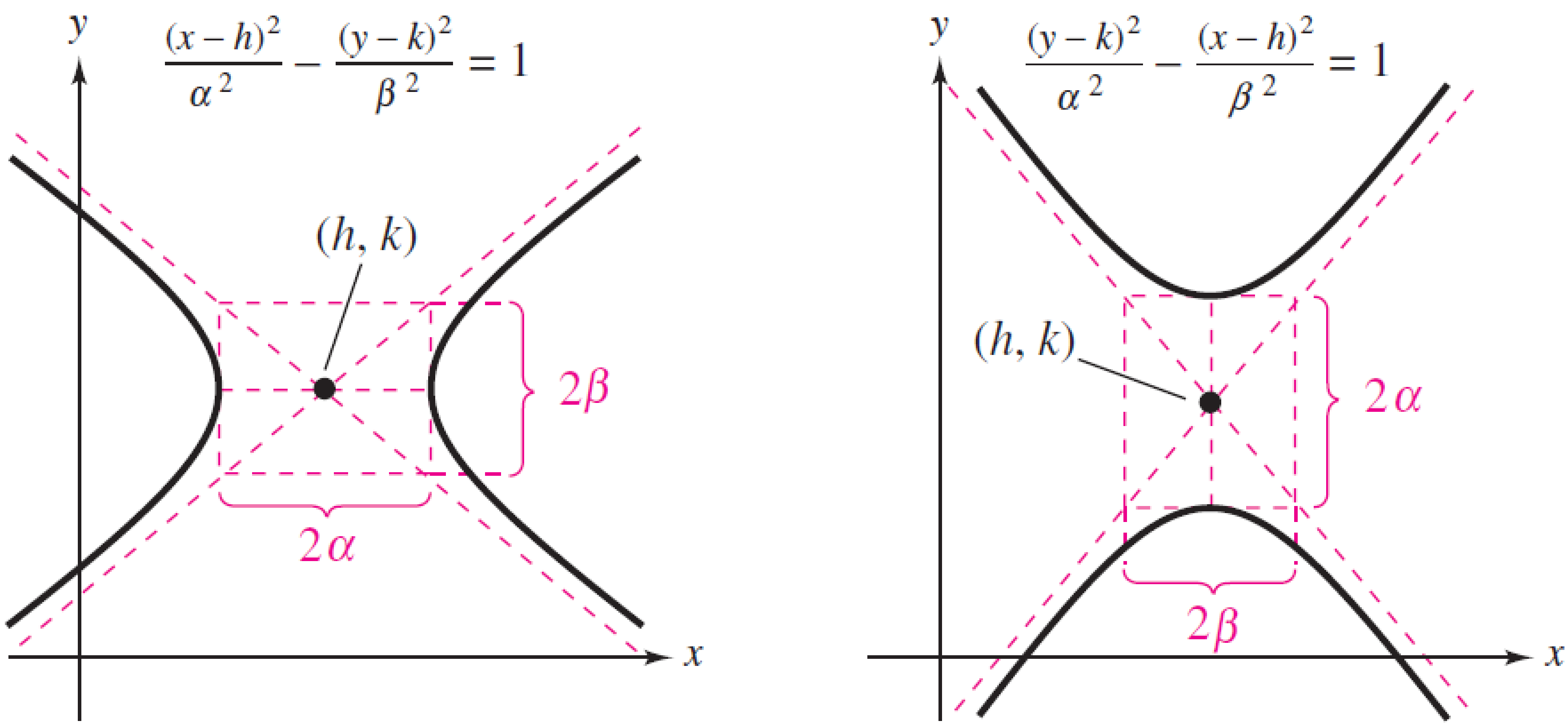
Circle (r = radius): $(x-h)^2+(y-k)^2=r^2$

Ellipse (2α = major axis length, 2β = minor axis length):

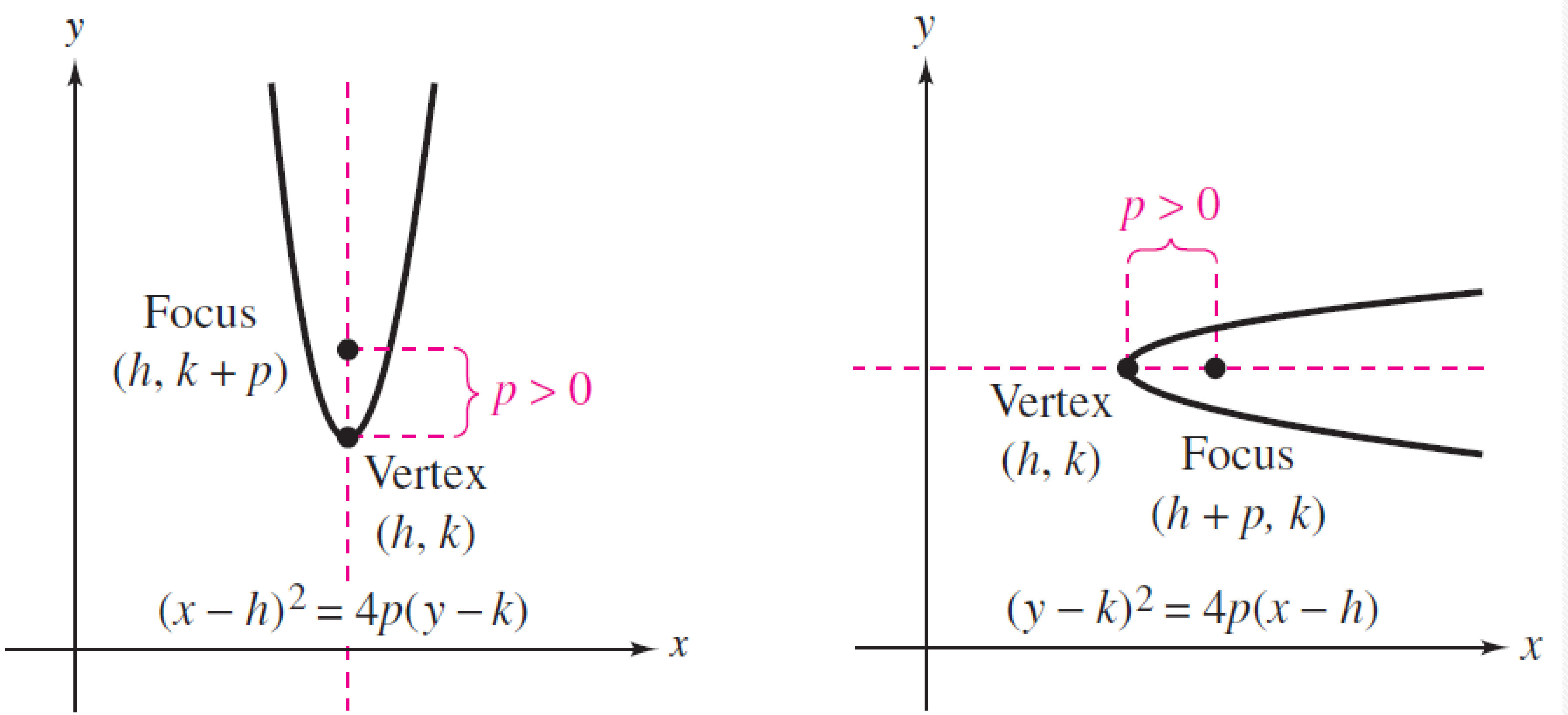




Hyperbola (2α = transverse axis length, 2β = conjugate axis length):



Parabola (p = directed distance from vertex to focus):



- Ex 5: (Identifying Conic Sections)

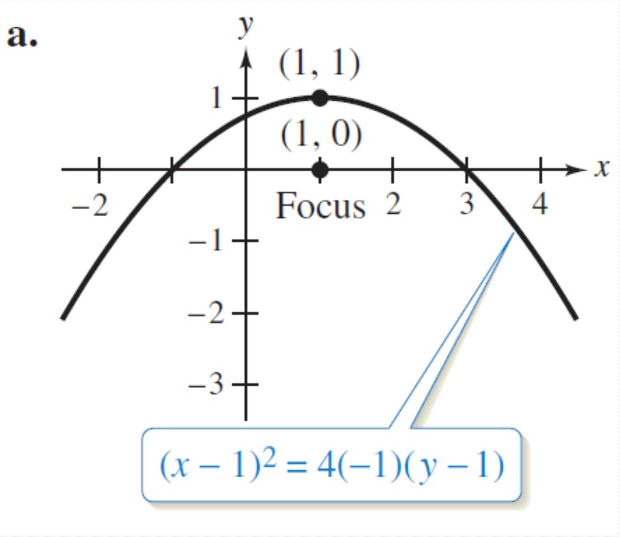
The standard form of $x^2 - 2x + 4y - 3 = 0$ is

$$(x-1)^2 = 4(-1)(y-1)$$

vertex at $(h, k) = (1, 1)$

The axis of the parabola is vertical. Because $p = -1$, the focus is the point (1, 0).

Because the focus lies below the vertex, the parabola opens downward.



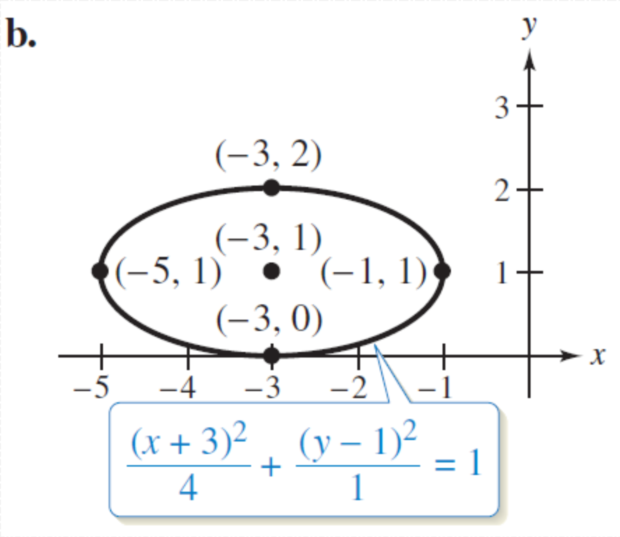
The standard form of $x^2 + 4y^2 + 6x - 8y + 9 = 0$ is

$$\frac{(x+3)^2}{4} + \frac{(y-1)^2}{1} = 1$$

center at $(h, k) = (-3, 1)$

$2\alpha = 4 \qquad 2\beta = 2$

The vertices of this ellipse occur at (−5, 1) and (−1, 1), and the end points of the minor axis occur at (−3, 2) and (−3, 0).



For second-degree equations that have an *xy*-term, the axes of the graphs of the corresponding conics are not parallel to the coordinate axes.

The required rotation θ angle (measured counterclockwise) is

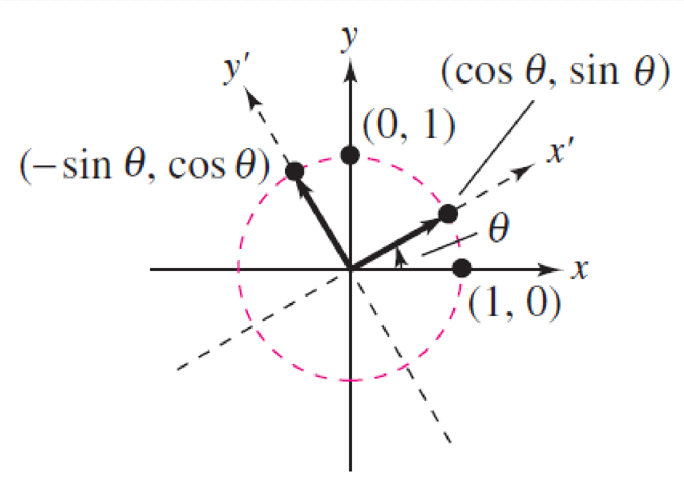
$$\cot 2\theta = \frac{a-c}{b}$$

standard basis

$$B = \{(1, 0), (0, 1)\}$$

new basis

$$B' = \{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\}$$



- Ex 6: (A Transition Matrix for Rotation in R^2)

$$B' = \{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\}$$

Sol: By Theorem 4.21

$$[B'\ B] = \begin{bmatrix} \cos\theta & -\sin\theta & 1 & 0 \\ \sin\theta & \cos\theta & 0 & 1 \end{bmatrix}$$

$$[I\ P^{-1}] = \begin{bmatrix} 1 & 0 & \cos\theta & \sin\theta \\ 0 & 1 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$x' = x\cos\theta + y\sin\theta \qquad y' = -x\sin\theta + y\cos\theta$$

$$x = x'\cos\theta - y'\sin\theta \qquad y = x'\sin\theta + y'\cos\theta$$

- Rotation of axes:

The general second-degree equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

can be written in the form

$$a'(x')^2 + c'(y')^2 + d'x' + e'y' + f' = 0$$

by rotating the coordinate axes counterclockwise through the angle θ, where θ is defined by

$$\cot 2\theta = \frac{a-c}{b}$$

The coefficients of the new equation are obtained from the substitutions

$$x = x'\cos\theta - y'\sin\theta \quad \text{and} \quad y = x'\sin\theta + y'\cos\theta$$

- Ex 7: (Rotation of a Conic Section)

Perform a rotation of axes to eliminate the *xy*-term in

$$5x^2-6xy+5y^2+14\sqrt{2}x-2\sqrt{2}y+18=0$$

and sketch the graph of the resulting equation in the *x'y'*-plane.

Sol: The angle of rotation

$$\cot 2\theta=\frac{a-c}{b}=\frac{5-5}{-6}=0$$

$$\theta=\pi/4$$

$$\sin\theta=\frac{1}{\sqrt{2}}\ \text{ and }\ \cos\theta=\frac{1}{\sqrt{2}}$$

By substituting

$$x=x'\cos\theta-y'\sin\theta=\frac{1}{\sqrt{2}}(x'-y')$$

$$y=x'\sin\theta+y'\cos\theta=\frac{1}{\sqrt{2}}(x'+y')$$

$$(x')^2 + 4(y')^2 + 6x' - 8y' + 9 = 0$$

$$\frac{(x'+3)^2}{2^2} + \frac{(y'-1)^2}{1^2} = \frac{(x'+3)^2}{4} + \frac{(y'-1)^2}{1} = 1$$

new basis

$$B' = \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}$$

the vertices

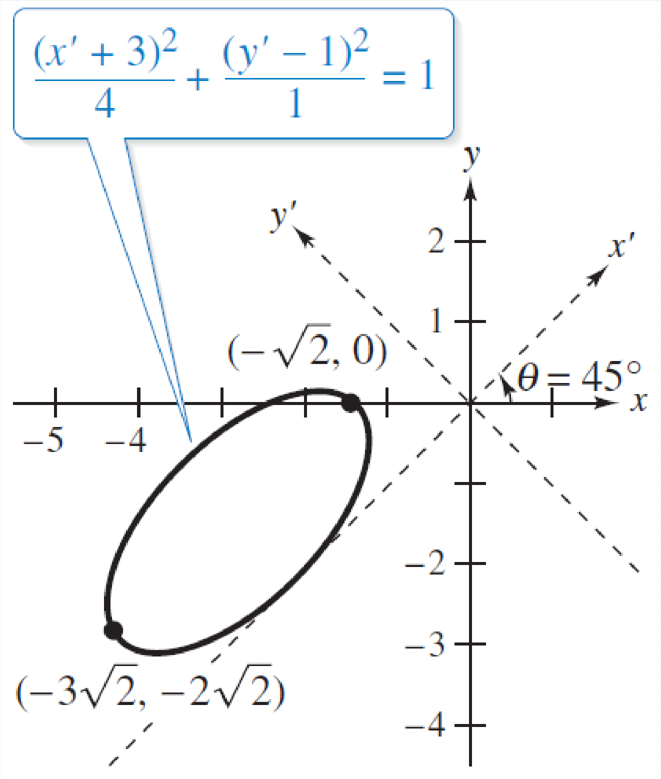
$$\begin{bmatrix} -5 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

use the equations

$$x = \frac{1}{\sqrt{2}}(x' - y') \text{ and } y = \frac{1}{\sqrt{2}}(x' + y')$$

obtain

$$(-3\sqrt{2}, -2\sqrt{2}) \text{ and } (-\sqrt{2}, 0)$$



Key Learning in Section 4.8

- Use the Wronskian to test a set of solutions of a linear homogeneous differential equation for linear independence.
- Identify and sketch the graph of a conic section and perform a rotation of axes.

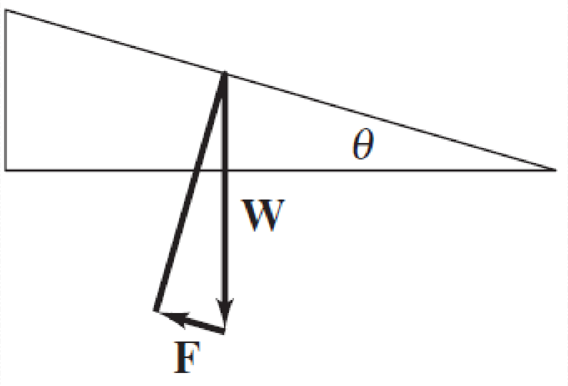
Keywords in Section 4.8

- Ellipse: 橢圓
- Hyperbola: 雙曲線
- Parabola: 拋物線
- Vertex: 頂點
- Focus: 焦點

- Force

Vectors have a wide variety of applications in engineering and the physical sciences. For example, to determine the amount of force required to pull an object up a ramp that has an angle of elevation θ, use the figure at the right.

In the figure, the vector labeled **W** represents the weight of the object, and the vector labeled **F** represents the required force. Using similar triangles and some trigonometry, the required force is **F** = **W** sin *θ*. (Verifying this.)



4.2 Linear Algebra Applied

In a mass-spring system, motion is assumed to occur in only the vertical direction. That is, the system has one *degree of freedom*. When the mass is pulled downward and then released, the system will oscillate. If the system is undamped, meaning that there are no forces present to slow or stop the oscillation, then the system will oscillate indefinitely. Applying Newton's Second Law of Motion to the mass yields the second order differential equation

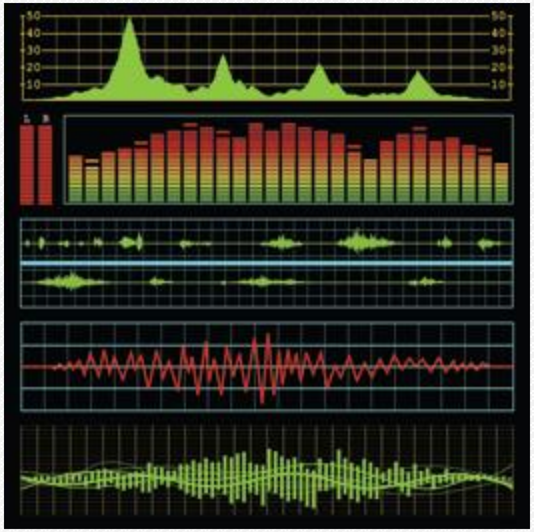
$$x'' + \omega^2 x = 0$$

where x is the displacement at time t and ω is a fixed constant called the *natural frequency* of the system. The general solution of this differential equation is

$$x(t) = a_1 \sin \omega t + a_2 \cos \omega t$$

where a_1 and a_2 are arbitrary constants. (Try verifying this.) In Exercise 41, you are asked to show that the set of all functions $x(t)$ is a vector space.

- Digital Sampling

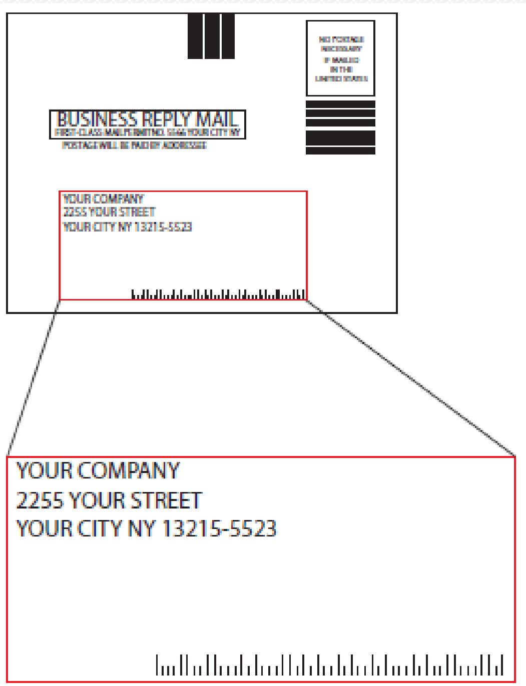


Digital signal processing depends on sampling, which converts continuous signals into discrete sequences that can be used by digital devices. Traditionally, sampling is uniform and pointwise, and is obtained from a single vector space. Then, the resulting sequence is reconstructed into a continuous-domain signal. Such a process, however, can involve a significant reduction in information, which could result in a low-quality reconstructed signal. In applications such as radar, geophysics, and wireless communications, researchers have determined situations in which sampling from a union of vector subspaces can be more appropriate.

- Image Morphing

Image morphing is the process of transforming one image into another by generating a sequence of synthetic intermediate images. Morphing has a wide variety of applications, such as movie special effects, age progression software, and simulating wound healing and cosmetic surgery results. Morphing an image uses a process called warping, in which a piece of an image is distorted. The mathematics behind warping and morphing can include forming a linear combination of the vectors that bound a triangular piece of an image, and performing an *affine transformation* to form new vectors and a distorted image piece.

- The RGB color model combinations of the colors red (**r**), green (**g**), and blue (**b**), known as the primary additive colors, to create all other colors in a system. Using the standard basis for R^3, where $\mathbf{r} = (1, 0, 0)$, $\mathbf{g} = (0, 1, 0)$ and $\mathbf{b} = (0, 0, 1)$ any visible color can be represented as a linear combination $c_1\mathbf{r} + c_2\mathbf{g} + c_3\mathbf{b}$ of the primary additive colors. The coefficients c_i are values between 0 and a specified maximum a inclusive. When $c_1 = c_2 = c_3$ the color is *grayscale*, with $c_i = 0$ representing black and $c_i = a$ representing white. The RGB color model is commonly used in computers, smart phones, televisions, and other electronic with a color display .

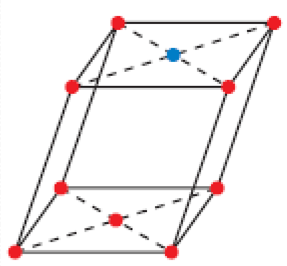


The U.S. Postal Service uses barcodes to represent such information as ZIP codes and delivery addresses. The ZIP + 4 barcode shown at the left starts with a long bar, has a series of short and long bars to represent each digit in the ZIP + 4 code and an additional digit for error checking, and ends with a long bar. The following is the code for the digits.

The error checking digit is such that when it is summed with the digits in the ZIP + 4 code, the result is a multiple of 10. (Verify this, as well as whether the ZIP + 4 code shown is coded correctly.) More sophisticated barcodes will also include error correcting digit(s). In an analogous way, matrices can be used to check for errors in transmitted messages. Information in the form of column vectors can be multiplied by an error detection matrix. When the resulting product is in the nullspace of the error detection matrix, no error in transmission exists. Otherwise, an error exists somewhere in the message. If the error detection matrix also has error correction, then the resulting matrix.

- Crystallography

Crystallography is the science of atomic and molecular structure. In a crystal, atoms are in a repeating pattern called a *lattice*. The simplest repeating unit in a lattice is a *unit cell*. Crystallographers can use bases and coordinate matrices in R^3 to designate the locations of atoms in a unit cell. For example, the figure below shows the unit cell known as *end-centered monoclinic*.



One possible coordinate matrix for the top end-centered (blue) atom is

$$[\mathbf{x}]_{B'} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}^T.$$

- Satellite Dish

A satellite dish is an antenna that is designed to transmit or receive signals of a specific type. A standard satellite dish consists of a bowl-shaped surface and a feed horn that is aimed toward the surface. The bowl-shaped surface is typically in the shape of an elliptic paraboloid. (See Section 7.4.) The cross section of the surface is typically in the shape of a rotated parabola.