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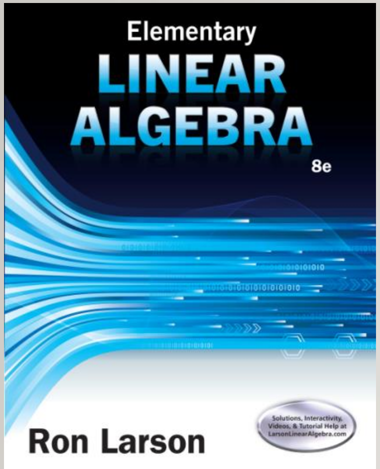

CHAPTER 6
LINEAR TRANSFORMATIONS

投影片設計製作者
淡江大學 電機系 翁慶昌 教授

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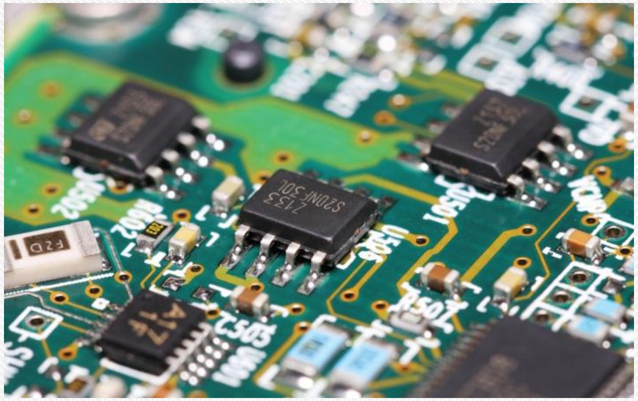

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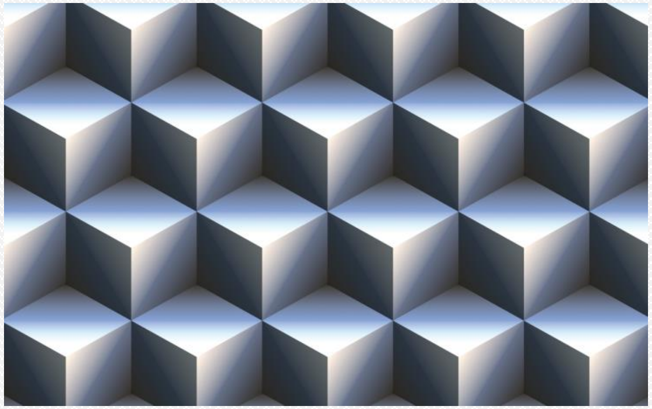

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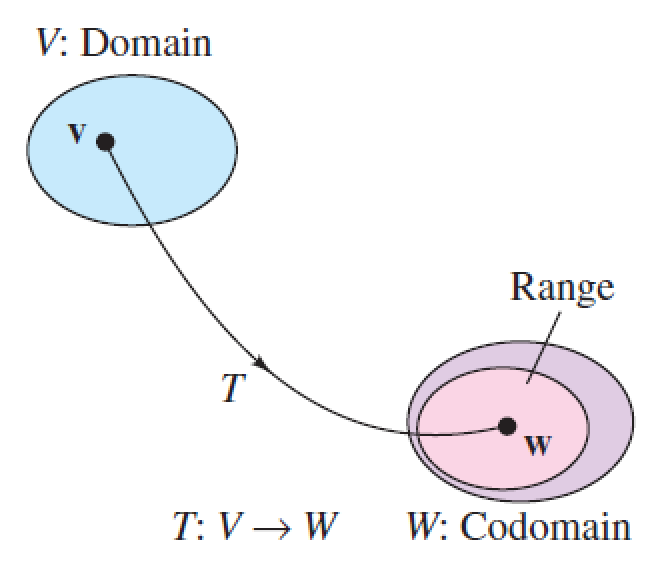
6.1 Introduction to Linear Transformations

- Function T that maps a vector space V into a vector space W:

$$T : V \xrightarrow{\text{mapping}} W, \quad V, W : \text{vector space}$$

V: the domain of T

W: the codomain of T



- Image of $\mathbf{v}$ under T:

If $\mathbf{v}$ is in V and $\mathbf{w}$ is in W such that

$$T(\mathbf{v}) = \mathbf{w}$$

Then $\mathbf{w}$ is called the image of $\mathbf{v}$ under T.

- the range of T:

The set of all images of vectors in V.

- the preimage of $\mathbf{w}$:

The set of all $\mathbf{v}$ in V such that $T(\mathbf{v})=\mathbf{w}$.

- Ex 1: (A function from R^2 into R^2)

$$T: R^2 \rightarrow R^2 \qquad \mathbf{v} = (v_1, v_2) \in R^2$$

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

(a) Find the image of **v**=(-1,2). (b) Find the preimage of **w**=(-1,11)

Sol:

$$(a) \ \ \mathbf{v} = (-1, 2)$$

$$\Rightarrow T(\mathbf{v}) = T(-1, 2) = (-1-2, -1+2(2)) = (-3, 3)$$

$$(b) \ \ T(\mathbf{v}) = \mathbf{w} = (-1, 11)$$

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2) = (-1, 11)$$

$$\Rightarrow v_1 - v_2 = -1$$

$$v_1 + 2v_2 = 11$$

$\Rightarrow v_1 = 3, \ v_2 = 4$ Thus {(3, 4)} is the preimage of **w**=(-1, 11).

- Linear Transformation (L.T.):

$V, W:$ vector space

$T : V \rightarrow W:$ V to W linear transformation

$$(1)\ T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in V$$

$$(2)\ T(c\mathbf{u}) = cT(\mathbf{u}), \quad \forall c \in R$$

- Notes:

(1) A linear transformation is said to be operation preserving.

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \qquad T(c\mathbf{u}) = cT(\mathbf{u})$$

Addition in V | Addition in W | Scalar multiplication in V | Scalar multiplication in W

(2) A linear transformation $T: V \rightarrow V$ from a vector space into itself is called a **linear operator**.

- Ex 2: (Verifying a linear transformation T from R^2 into R^2)

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

Pf:

$\mathbf{u} = (u_1, u_2),\ \mathbf{v} = (v_1, v_2)$: vector in R^2, c : any real number

(1)Vector addition :

$$\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$$

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T(u_1 + v_1, u_2 + v_2) \\ &= ((u_1 + v_1) - (u_2 + v_2), (u_1 + v_1) + 2(u_2 + v_2)) \\ &= ((u_1 - u_2) + (v_1 - v_2), (u_1 + 2u_2) + (v_1 + 2v_2)) \\ &= (u_1 - u_2, u_1 + 2u_2) + (v_1 - v_2, v_1 + 2v_2) \\ &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

(2) Scalar multiplication

$$c\mathbf{u} = c(u_1, u_2) = (cu_1, cu_2)$$

$$\begin{aligned} T(c\mathbf{u}) &= T(cu_1, cu_2) = (cu_1 - cu_2, cu_1 + 2cu_2) \\ &= c(u_1 - u_2, u_1 + 2u_2) \\ &= cT(\mathbf{u}) \end{aligned}$$

Therefore, T is a linear transformation.

- Ex 3: (Functions that are not linear transformations)

(a) $f(x) = \sin x$

$$\sin(x_1 + x_2) \neq \sin(x_1) + \sin(x_2)$$

$$\sin(\tfrac{\pi}{2} + \tfrac{\pi}{3}) \neq \sin(\tfrac{\pi}{2}) + \sin(\tfrac{\pi}{3})$$

$\Leftarrow f(x) = \sin x$ is not linear transformation

(b) $f(x) = x^2$

$$(x_1 + x_2)^2 \neq x_1^2 + x_2^2$$

$$(1+2)^2 \neq 1^2 + 2^2$$

$\Leftarrow f(x) = x^2$ is not linear transformation

(c) $f(x) = x + 1$

$$f(x_1 + x_2) = x_1 + x_2 + 1$$

$$f(x_1) + f(x_2) = (x_1 + 1) + (x_2 + 1) = x_1 + x_2 + 2$$

$$f(x_1 + x_2) \neq f(x_1) + f(x_2)$$

$\Leftarrow f(x) = x + 1$ is not linear transformation

- Notes: Two uses of the term "linear".

(1) $f(x)=x+1$ is called a linear function because its graph is a line.

(2) $f(x)=x+1$ is not a linear transformation from a vector space R into R because it preserves neither vector addition nor scalar multiplication.

- Zero transformation:

$$T:V\rightarrow W \qquad\qquad T(\mathbf{v})=0,\ \ \forall\mathbf{v}\in V$$

- Identity transformation:

$$T:V\rightarrow V \qquad\qquad T(\mathbf{v})=\mathbf{v},\ \ \forall\mathbf{v}\in V$$

- Thm 6.1: (Properties of linear transformations)

$$T:V\rightarrow W,\quad \mathbf{u},\mathbf{v}\in V$$

$$(1)\ T(\mathbf{0})=\mathbf{0}$$

$$(2)\ T(-\mathbf{v})=-T(\mathbf{v})$$

$$(3)\ T(\mathbf{u}-\mathbf{v})=T(\mathbf{u})-T(\mathbf{v})$$

$$(4)\ \text{If } \mathbf{v}=c_1v_1+c_2v_2+\cdots+c_nv_n$$

$$\text{Then } T(\mathbf{v})=T(c_1v_1+c_2v_2+\cdots+c_nv_n)$$

$$=c_1T(v_1)+c_2T(v_2)+\cdots+c_nT(v_n)$$

- Ex 4: (Linear transformations and bases)

Let $T : R^3 \to R^3$ be a linear transformation such that

$$T(1,0,0) = (2,-1,4)$$
$$T(0,1,0) = (1,5,-2)$$
$$T(0,0,1) = (0,3,1)$$

Find $T(2, 3, -2)$.

Sol:

$$(2,3,-2) = 2(1,0,0) + 3(0,1,0) - 2(0,0,1)$$

$$T(2,3,-2) = 2T(1,0,0) + 3T(0,1,0) - 2T(0,0,1) \qquad (T \text{ is a L.T.})$$
$$= 2(2,-1,4) + 3(1,5,-2) - 2T(0,3,1)$$
$$= (7,7,0)$$

- Ex 5: (A linear transformation defined by a matrix)

The function $T: R^2 \rightarrow R^3$ is defined as $T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

(a) Find $T(\mathbf{v})$, where $\mathbf{v} = (2,-1)$

(b) Show that T is a linear transformation form R^2 into R^3

Sol: (a) $\mathbf{v} = (2,-1)$

R^2 vector ↓ R^3 vector ↓

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

$$\therefore T(2,-1) = (6,3,0)$$

(b) $T(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$ (vector addition)

$T(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cT(\mathbf{u})$ (scalar multiplication)

- Thm 6.2: (The linear transformation given by a matrix)

Let A be an $m \times n$ matrix. The function T defined by

$$T(\mathbf{v}) = A\mathbf{v}$$

is a linear transformation from R^n into R^m.

- Note:

R^n vector $\quad$ R^m vector

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}$$

$$T(\mathbf{v}) = A\mathbf{v}$$

$$T : R^n \longrightarrow R^m$$

- Ex 7: (Rotation in the plane)

Show that the L.T. $T : R^2 \to R^2$ given by the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

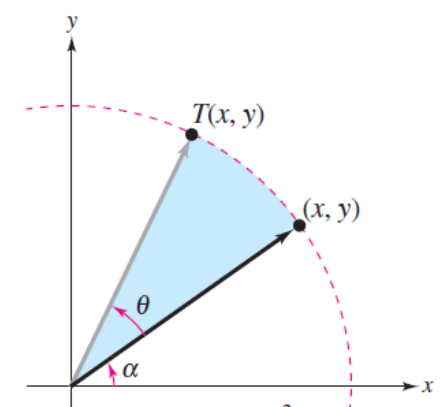
has the property that it rotates every vector in R^2 counterclockwise about the origin through the angle θ.

Sol:

$v = (x, y) = (r\cos\alpha, r\sin\alpha)$ (polar coordinates)

r : the length of v

α : the angle from the positive x-axis counterclockwise to the vector v



Rotation in R^2

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\ r\sin\theta\cos\alpha + r\cos\theta\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} r\cos(\theta+\alpha) \\ r\sin(\theta+\alpha) \end{bmatrix}$$

r ： the length of $T(\mathbf{v})$

$\theta+\alpha$ ： the angle from the positive x-axis counterclockwise to the vector $T(\mathbf{v})$

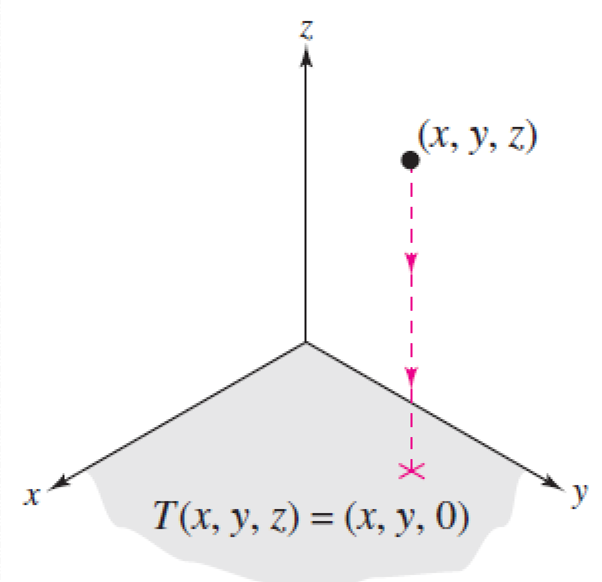
Thus, $T(\mathbf{v})$ is the vector that results from rotating the vector $\mathbf{v}$ counterclockwise through the angle θ.

- Ex 8: (A projection in R^3)

The linear transformation $T: R^3 \rightarrow R^3$ is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is called a projection in R^3.



Projection onto xy-plane

- Ex 9: (A linear transformation from $M_{m\times n}$ into $M_{n\times m}$)

$$T(A)=A^T \quad (T: M_{m\times n}\rightarrow M_{n\times m})$$

Show that T is a linear transformation.

Sol:

$$A, B \in M_{m\times n}$$

$$T(A+B)=(A+B)^T=A^T+B^T=T(A)+T(B)$$

$$T(cA)=(cA)^T=cA^T=cT(A)$$

Therefore, T is a linear transformation from $M_{m\times n}$ into $M_{n\times m}$.

Key Learning in Section 6.1

- Find the image and preimage of a function.
- Show that a function is a linear transformation, and find a linear transformation.

Keywords in Section 6.1

- function: 函數
- domain: 論域
- codomain: 對應論域
- image of v under T: 在T映射下v的像
- range of T: T的值域
- preimage of w: w的反像
- linear transformation: 線性轉換
- linear operator: 線性運算子
- zero transformation: 零轉換
- identity transformation: 相等轉換

6.2 The Kernel and Range of a Linear Transformation

- Kernel of a linear transformation T:

Let $T: V \to W$ be a linear transformation

Then the set of all vectors **v** in V that satisfy $T(\mathbf{v}) = 0$ is called the kernel of T and is denoted by ker(T).

$$\ker(T) = \{\mathbf{v} \mid T(\mathbf{v}) = 0, \forall \mathbf{v} \in V\}$$

- Ex 1: (Finding the kernel of a linear transformation)

$$T(A) = A^T \quad (T: M_{3\times2} \to M_{2\times3})$$

Sol:

$$\ker(T) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

- Ex 2: (The kernel of the zero and identity transformations)

(*a*) $T(\mathbf{v}) = \mathbf{0}$ (the zero transformation $T: V \rightarrow W$)

$$\ker(T) = V$$

(*b*) $T(\mathbf{v}) = \mathbf{v}$ (the identity transformation $T: V \rightarrow V$)

$$\ker(T) = \{\mathbf{0}\}$$

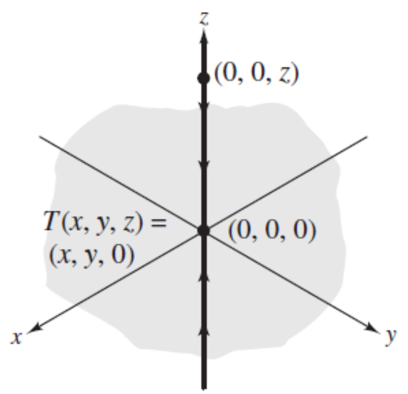
- Ex 3: (Finding the kernel of a linear transformation)

$$T(x, y, z) = (x, y, 0) \qquad (T: R^3 \rightarrow R^3)$$

$$\ker(T) = ?$$

Sol:

$$\ker(T) = \{(0, 0, z) \mid z \text{ is a real number }\}$$



The kernel of T is the set of all vectors on the z-axis.

- Ex 5: (Finding the kernel of a linear transformation)

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (T: R^3 \rightarrow R^2)$$

$$\ker(T) = ?$$

Sol:

$$\ker(T) = \{(x_1, x_2, x_3) \mid T(x_1, x_2, x_3) = (0,0),\ x = (x_1, x_2, x_3) \in R^3\}$$

$$T(x_1, x_2, x_3) = (0,0)$$

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -2 & 0 \\ -1 & 2 & 3 & 0 \end{bmatrix} \xrightarrow{G.J.E} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \ker(T) = \{t(1,-1,1) \mid t \text{ is a real number }\}$$

$$= \text{span}\{(1,-1,1)\}$$

- Thm 6.3: (The kernel is a subspace of *V*)

  The kernel of a linear transformation $T: V \to W$ is a subspace of the domain *V*.

Pf: $\because T(\mathbf{0}) = \mathbf{0}$ (Theorem 6.1)

$\therefore \ker(T)$ is a nonempty subset of V

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in the kernel of T. then

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{0}+\mathbf{0} = \mathbf{0} \quad \Rightarrow \mathbf{u}+\mathbf{v} \in \ker(T)$$

$$T(c\mathbf{u}) = cT(\mathbf{u}) = c\mathbf{0} = \mathbf{0} \quad \Rightarrow c\mathbf{u} \in \ker(T)$$

Thus, $\ker(T)$ is a subspace of V.

- Note:

  The kernel of *T* is sometimes called the **nullspace** of *T*.

- Ex 6: (Finding a basis for the kernel)

Let $T: R^5 \to R^4$ be defined by $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x}$ is in R^5 and

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

Find a basis for ker(T) as a subspace of R^5.

Sol:

$$[A \mid 0] =$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & -1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ -1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 8 & 0 \end{bmatrix} \xrightarrow{G.J.E} \begin{bmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

s t

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s+t \\ s+2t \\ s \\ -4t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

$B = \{(-2, 1, 1, 0, 0), (1, 2, 0, -4, 1)\}$: one basis for the kernel of T

- **Corollary to Thm 6.3:**

Let $T: R^n \rightarrow R^m$ be the L.T given by $T(\mathbf{x}) = A\mathbf{x}$

Then the kernel of T is equal to the solution space of $A\mathbf{x} = 0$

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{(a linear transformation } T: R^n \rightarrow R^m\text{)}$$

$$\Rightarrow Ker(T) = NS(A) = \left\{\mathbf{x} \mid A\mathbf{x} = 0, \forall \mathbf{x} \in R^m\right\} \text{ (subspace of } R^m\text{)}$$

- **Range of a linear transformation *T*:**

Let $T: V \rightarrow W$ be a L.T.

Then the set of all vectors w in W that are images of vector in V is called the range of T and is denoted by $range(T)$

$$range(T) = \{T(\mathbf{v}) \mid \forall \mathbf{v} \in V\}$$

- Thm 6.4: (The range of *T* is a subspace of *W*)

The range of a linear transformation $T: V \to W$ is a subspace of W.

Pf:

$\because T(\mathbf{0}) = \mathbf{0}$ (Thm.6.1)

$\therefore range(T)$ is a nonempty subset of W

Let $T(\mathbf{u})$ and $T(\mathbf{v})$ be vector in the range of T

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \in range(T)$ $(\mathbf{u} \in V, \mathbf{v} \in V \Rightarrow \mathbf{u} + \mathbf{v} \in V)$

$T(c\mathbf{u}) = cT(\mathbf{u}) \in range(T)$ $(\mathbf{u} \in V \Rightarrow c\mathbf{u} \in V)$

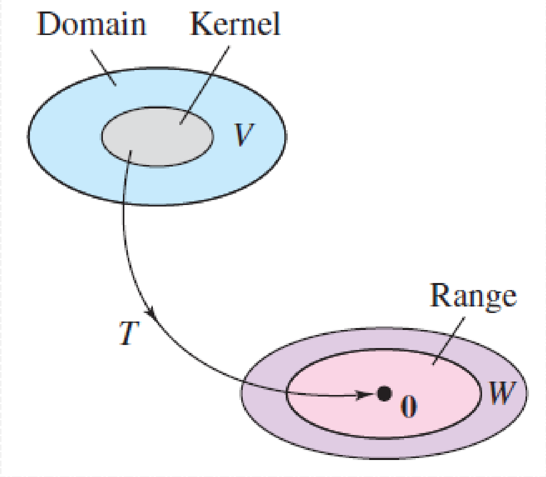
Therefore, $range(T)$ is W subspace.

- Notes:

$$T: V \to W \text{ is a L.T.}$$

(1) $Ker(T)$ is subspace of V

(2) $range(T)$ is subspace of W



- Corollary to Thm 6.4:

Let $T: R^n \to R^m$ be the L.T. given by $T(\mathbf{x}) = A\mathbf{x}$

Then the range of T is equal to the column space of A

$\Rightarrow \text{range}(T) = CS(A)$

- Ex 7: (Finding a basis for the range of a linear transformation)

Let $T : R^5 \rightarrow R^4$ be defined by $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x}$ is R^5 and

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

Find a basis for the range of T.

Sol:

$$A=\begin{bmatrix}1&2&0&1&-1\\2&1&3&1&0\\-1&0&-2&0&1\\0&0&0&2&8\end{bmatrix}\xrightarrow{G.J.E}\begin{bmatrix}1&0&2&0&-1\\0&1&-1&0&2\\0&0&0&1&4\\0&0&0&0&0\end{bmatrix}=B$$

$$\begin{matrix}c_1&c_2&c_3&c_4&c_5\end{matrix}\qquad\qquad\begin{matrix}w_1&w_2&w_3&w_4&w_5\end{matrix}$$

$\Rightarrow \{w_1, w_2, w_4\}$ is a basis for $CS(B)$

$\{c_1, c_2, c_4\}$ is a basis for $CS(A)$

$\Rightarrow \{(1, 2, -1, 0), (2, 1, 0, 0), (1, 1, 0, 2)\}$ is a basis for the range of T

- Rank of a linear transformation $T:V\rightarrow W$:

$$rank(T) = \text{the dimension of the range of } T$$

- Nullity of a linear transformation $T:V\rightarrow W$:

$$nullity(T) = \text{the dimension of the kernel of } T$$

- Note:

Let $T: R^n \rightarrow R^m$ be the L.T. given by $T(\mathbf{x}) = A\mathbf{x}$, then

$$rank(T) = rank(A)$$

$$nullity(T) = nullity(A)$$

- Thm 6.5: (Sum of rank and nullity)

Let $T: V \rightarrow W$ be a L.T. form an n-dimensional vector space V into a vector space W. then

$$\text{rank}(T) + \text{nullity}(T) = n$$

$$\dim(\text{range of } T) + \dim(\text{kernel of } T) = \dim(\text{domain of } T)$$

Pf:

Let T is represented by an $m \times n$ $matrix$ A

Assume $rank(A) = r$

(1) $\text{rank}(T) = \dim(\text{range of } T) = \dim(\text{column space of } A)$

$= \text{rank}(A) = r$

(2) $\text{nullity}(T) = \dim(\text{kernel of } T) = \dim(\text{solution space of } A)$

$= n - r$

$$\Rightarrow rank(T) + nullity(T) = r + (n - r) = n$$

- Ex 8: (Finding the rank and nullity of a linear transformation)

Find the rank and nullity of the L.T. $T: R^3 \rightarrow R^3$ define by

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Sol:

$$\text{rank}(T) = \text{rank}(A) = 2$$

$$\text{nullity}(T) = \dim(\text{domain of } T) - \text{rank}(T) = 3 - 2 = 1$$

- Ex 9: (Finding the rank and nullity of a linear transformation)

Let $T: R^5 \rightarrow R^7$ be a linear transformation.

(a) Find the dimension of the kernel of T if the dimension of the range is 2

(b) Find the rank of T if the nullity of T is 4

(c) Find the rank of T if $\ker(T) = \{0\}$

Sol:

(a) $\dim(\text{domain of } T) = 5$

$$\dim(\text{kernel of } T) = n - \dim(\text{range of } T) = 5 - 2 = 3$$

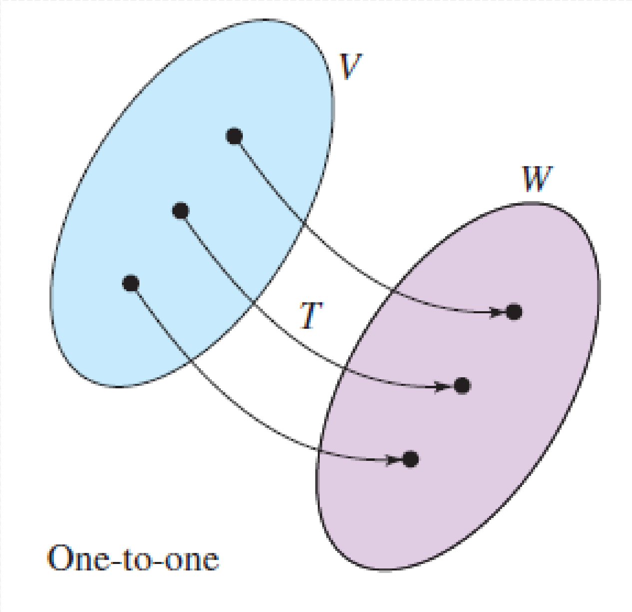
(b) $\text{rank}(T) = n - \text{nullity}(T) = 5 - 4 = 1$

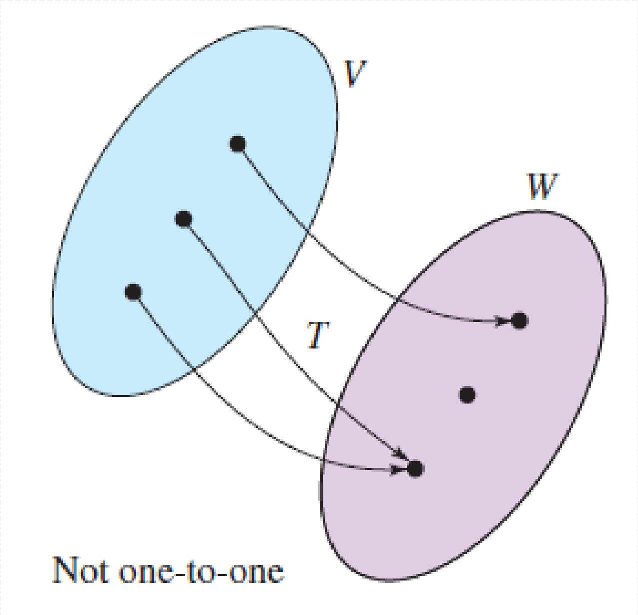
(c) $\text{rank}(T) = n - \text{nullity}(T) = 5 - 0 = 5$

- One-to-one:

A function $T:V \to W$ is called one-to-one if the preimage of every w in the range consists of a single vector.

T is one-to-one iff for all u and v in V, $T(\mathbf{u}) = T(\mathbf{v})$ implies that $\mathbf{u} = \mathbf{v}$.





- Onto:

A function $T: V \rightarrow W$ is said to be onto if every element in $\mathbf{w}$ has a preimage in V

(T is onto W when W is equal to the range of T.)

- Thm 6.6: (One-to-one linear transformation)

Let $T: V \to W$ be a L.T.

Then T is 1-1 iff $\ker(T) = \{0\}$

Pf:

Suppose T is 1-1

Then $T(v) = 0$ can have only one solution $: v = 0$

i.e. $\ker(T) = \{0\}$

Suppose $\ker(T) = \{0\}$ and $T(u) = T(v)$

$T(u - v) = T(u) - T(v) = 0$

T is a L.T.

$\because u - v \in \ker(T) \Rightarrow u - v = 0$

$\Rightarrow T$ is 1-1

- Ex 10: (One-to-one and not one-to-one linear transformation)

(*a*) The L.T. $T: M_{m\times n} \rightarrow M_{n\times m}$ given by $T(A) = A^T$ is one-to-one.

Because its kernel consists of only the $m\times n$ zero matrix.

(*b*) The zero transformation $T: R^3 \rightarrow R^3$ is not one-to-one.

Because its kernel is all of R^3.

- Thm 6.7: (Onto linear transformation)

Let $T:V \rightarrow W$ be a L.T., where W is finite dimensional.

Then T is onto iff the rank of T is equal to the dimension of W.

- Thm 6.8: (One-to-one and onto linear transformation)

Let $T:V \rightarrow W$ be a L.T. with vector space V and W both of dimension n. Then T is one-to-one if and only if it is onto.

Pf:

If T is one-to-one, then $\ker(T)=\{0\}$ and $\dim(\ker(T))=0$

$$\dim(\text{range}(T)) = n - \dim(\ker(T)) = n = \dim(W)$$

Consequently, T is onto.

If T is onto, then $\dim(\text{range of } T) = \dim(W) = n$

$$\dim(\ker(T)) = n - \dim(\text{range of } T) = n - n = 0$$

Therefore, T is one-to-one.

- Ex 11:

The L.T. $T: R^n \rightarrow R^m$ is given by $T(\mathbf{x}) = A\mathbf{x}$, Find the nullity and rank of T and determine whether T is one-to-one, onto, or neither.

$$(a) A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad (b) A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$(c) A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \qquad (d) A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Sol:

$T:R^n \rightarrow R^m$	dim(domain of T)	rank(T)	nullity(T)	1-1	onto
$(a) T:R^3 \rightarrow R^3$	3	3	0	Yes	Yes
$(b) T:R^2 \rightarrow R^3$	2	2	0	Yes	No
$(c) T:R^3 \rightarrow R^2$	3	2	1	No	Yes
$(d) T:R^3 \rightarrow R^3$	3	2	1	No	No

- Isomorphism:

A linear transformation $T: V \to W$ that is one to one and onto is called an isomorphism. Moreover, if V and W are vector spaces such that there exists an isomorphism from V to W, then V and W are said to be isomorphic to each other.

- Thm 6.9: (Isomorphic spaces and dimension)

Two finite-dimensional vector space V and W are isomorphic if and only if they are of the same dimension.

Pf:

Assume that V is isomorphic to W, where V has dimension n.

$\Rightarrow$ There exists a L.T. $T: V \to W$ that is one to one and onto.

$\because T$ is one-to-one

$\Rightarrow \dim(Ker(T)) = 0$

$\Rightarrow \dim(\text{range of } T) = \dim(\text{domain of } T) - \dim(Ker(T)) = n - 0 = n$

$\because T$ is onto.

$\Rightarrow \dim(\text{range of } T) = \dim(W) = n$

Thus $\dim(V) = \dim(W) = n$

Assume that V and W both have dimension n.

Let $\{v_1, v_2, \cdots, v_n\}$ be a basis of V, and

let $\{w_1, w_2, \cdots, w_n\}$ be a basis of W.

Then an arbitrary vector in V can be represented as

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

and you can define a L.T. $T : V \to W$ as follows.

$$T(\mathbf{v}) = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_n\mathbf{w}_n$$

It can be shown that this L.T. is both 1-1 and onto.

Thus V and W are isomorphic.

- Ex 12: (Isomorphic vector spaces)

The following vector spaces are isomorphic to each other.

(a) $R^4 = 4\text{-space}$

(b) $M_{4\times1} = \text{space of all } 4\times1 \text{ matrices}$

(c) $M_{2\times2} = \text{space of all } 2\times2 \text{ matrices}$

(d) $P_3(x) = \text{space of all polynomials of degree 3 or less}$

(e) $V = \{(x_1, x_2, x_3, x_4, 0), x_i \text{ is a real number}\}\text{(subspace of } R^5)$

Key Learning in Section 6.2

- Find the kernel of a linear transformation.
- Find a basis for the range, the rank, and the nullity of a linear transformation.
- Determine whether a linear transformation is one-to-one or onto.
- Determine whether vector spaces are isomorphic.

Keywords in Section 6.2

- kernel of a linear transformation T: 線性轉換T的核空間
- range of a linear transformation T: 線性轉換T的值域
- rank of a linear transformation T: 線性轉換T的秩
- nullity of a linear transformation T: 線性轉換T的核次數
- one-to-one: 一對一
- onto: 映成
- isomorphism(one-to-one and onto): 同構
- isomorphic space: 同構的空間

6.3 Matrices for Linear Transformations

- Two representations of the linear transformation $T: R^3 \to R^3$:

$$(1) T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$$

$$(2) T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- Three reasons for matrix representation of a linear transformation:
    - It is simpler to write.
    - It is simpler to read.
    - It is more easily adapted for computer use.

- Thm 6.10: (Standard matrix for a linear transformation)

Let $T: R^n \rightarrow R^m$ be a linear trt ansformati on such that

$$T(e_1)=\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad T(e_2)=\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \cdots, \quad T(e_n)=\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

Then the $m \times n$ matrix who se n columns correspond to $T(e_i)$

$$A=\left[T(e_1) \mid T(e_2) \mid \cdots \mid T(e_n)\right]=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $T(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v}$ in R^n.

A is called the standard matrix for T.

Pf:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 e_1 + v_2 e_2 + \cdots + v_n e_n$$

$$T \text{ is a L.T.} \Rightarrow T(\mathbf{v}) = T(v_1 e_1 + v_2 e_2 + \cdots + v_n e_n)$$

$$= T(v_1 e_1) + T(v_2 e_2) + \cdots + T(v_n e_n)$$

$$= v_1 T(e_1) + v_2 T(e_2) + \cdots + v_n T(e_n)$$

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}$$

$$= v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= v_1 T(e_1) + v_2 T(e_2) + \cdots + v_n T(e_n)$$

Therefore, $T(\mathbf{v}) = A\mathbf{v}$ for each $\mathbf{v}$ in R^n

- Ex 1: (Finding the standard matrix of a linear transformation)

Find the standard matrix for the L.T. $T: R^3 \rightarrow R^2$ define by

$$T(x, y, z) = (x - 2y, 2x + y)$$

Sol:

Vector Notation

$$T(e_1) = T(1, 0, 0) = (1, 2)$$

$$T(e_2) = T(0, 1, 0) = (-2, 1)$$

$$T(e_3) = T(0, 0, 1) = (0, 0)$$

Matrix Notation

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$T(e_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A = \left[T(e_1) \mid T(e_2) \mid T(e_3) \right]$$

$$= \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

- Check:

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-2y \\ 2x+y \end{bmatrix}$$

$$\text{i.e. } T(x, y, z) = (x-2y, 2x+y)$$

- Note:

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{matrix} \leftarrow & 1x - 2y + 0z \\ \leftarrow & 2x + 1y + 0z \end{matrix}$$

- Ex 2: (Finding the standard matrix of a linear transformation)

The linear transformation $T: R^2 \rightarrow R^2$ is given by projecting each point in R^2 onto the x-axis. Find the standard matrix for T.

Sol:

$$T(x, y) = (x, 0)$$

$$A = \left[T(e_1) \mid T(e_2)\right] = \left[T(1,0) \mid T(0,1)\right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

- Notes:

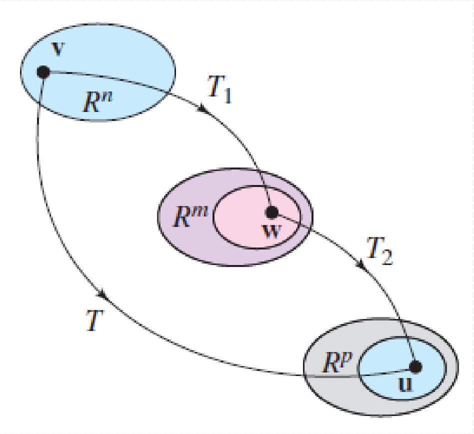
(1) The standard matrix for the zero transformation from R^n into R^m is the $m \times n$ zero matrix.

(2) The standard matrix for the zero transformation from R^n into R^n is the $n \times n$ identity matrix I_n.

- Composition of $T_1:R^n \to R^m$ with $T_2:R^m \to R^p$:

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})), \quad \mathbf{v} \in R^n$$

$$T = T_2 \circ T_1, \quad \text{domain of } T = \text{domain of } T_1$$



- Thm 6.11: (Composition of linear transformations)

Let $T_1 : R^n \to R^m$ and $T_2 : R^m \to R^p$ be L.T.

with standard matrices A_1 and A_2, then

(1) The composition $T : R^n \to R^p$, defined by $T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$, is a L.T.

(2) The standard matrix A for T is given by the matrix product $A = A_2 A_1$

Pf:

(1)(T is a L.T.)

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in R^n and let c be any scalar the n

$$T(\mathbf{u}+\mathbf{v}) = T_2(T_1(\mathbf{u}+\mathbf{v})) = T_2(T_1(\mathbf{u})+T_1(\mathbf{v}))$$

$$= T_2(T_1(\mathbf{u})) + T_2(T_1(\mathbf{v})) = T(\mathbf{u}) + T(\mathbf{v})$$

$$T(c\mathbf{v}) = T_2(T_1(c\mathbf{v})) = T_2(cT_1(\mathbf{v})) = cT_2(T_1(\mathbf{v})) = cT(\mathbf{v})$$

(2)(A_2A_1 is the standard matrix for T)

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})) = T_2(A_1\mathbf{v}) = A_2A_1\mathbf{v} = (A_2A_1)\mathbf{v}$$

- Note:

$$T_1 \circ T_2 \neq T_2 \circ T_1$$

- Ex 3: (The standard matrix of a composition)

Let T_1 and T_2 be L.T. from R^3 into R^3 s.t.

$$T_1(x, y, z) = (2x + y, 0, x + z)$$

$$T_2(x, y, z) = (x - y, \text{z}, y)$$

Find the standard matrices for the compositio ns

$T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$,

Sol:

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{(standard matrix for } T_1\text{)}$$

$$A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{(standard matrix for } T_2\text{)}$$

The standard matrix for $T = T_2 \circ T_1$

$$A = A_2 A_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The standard matrix for $T' = T_1 \circ T_2$

$$A' = A_1 A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- Inverse linear transformation:

$$\text{If } T_1 : R^n \rightarrow R^n \text{ and } T_2 : R^n \rightarrow R^n \text{ are L.T. s.t. for every } \mathbf{v} \text{ in } R^n$$

$$T_2(T_1(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad T_1(T_2(\mathbf{v})) = \mathbf{v}$$

Then T_2 is called the inverse of T_1 and T_1 is said to be invertible

- Note:

If the transformation T is invertible, then the inverse is unique and denoted by T^{-1}.

- Thm 6.12: (Existence of an inverse transformation)

Let $T: R^n \rightarrow R^n$ be a L.T. with standard matrix A,
Then the following condition are equivalent.

(1) T is invertible.

(2) T is an isomorphism.

(3) A is invertible.

- Note:

If T is invertible with standard matrix A, then the standard matrix for T^{-1} is A^{-1}.

- Ex 4: (Finding the inverse of a linear transformation)

The L.T. $T: R^3 \rightarrow R^3$ is defined by

$$T(x_1, x_2, x_3) = (2x_1 + 3x_2 + x_3, 3x_1 + 3x_2 + x_3, 2x_1 + 4x_2 + x_3)$$

Show that T is invertible, and find its inverse.

Sol:

The standard matrix for T

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \quad \begin{matrix} \leftarrow 2x_1 + 3x_2 + x_3 \\ \leftarrow 3x_1 + 3x_2 + x_3 \\ \leftarrow 2x_1 + 4x_2 + x_3 \end{matrix}$$

$$\left[A \mid I_3\right] = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{G.J.E} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right] = \left[I \mid A^{-1}\right]$$

Therefore T is invertible and the standard matrix for T^{-1} is A^{-1}

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

$$T^{-1}(\mathbf{v}) = A^{-1}\mathbf{v} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ -x_1 + x_3 \\ 6x_1 - 2x_2 - 3x_3 \end{bmatrix}$$

In other words,

$$T^{-1}(x_1, x_2, x_3) = (-x_1 + x_2, -x_1 + x_3, 6x_1 - 2x_2 - 3x_3)$$

- the matrix of *T* relative to the bases *B and B'*:

$$T: V \rightarrow W \qquad \text{(a L.T.)}$$

$$B = \{v_1, v_2, \cdots, v_n\} \qquad \text{(a basis for } V\text{)}$$

$$B' = \{w_1, w_2, \cdots, w_m\} \qquad \text{(a basis for } W\text{)}$$

Thus, the matrix of *T* relative to the bases *B and B'* is

$$A = \left[[T(v_1)]_{B'}, [T(v_2)]_{B'}, \cdots, [T(v_n)]_{B'} \right] \in M_{m \times n}$$

- Transformation matrix for nonstandard bases:

Let V and W be finite-dimensional vector spaces with basis B and B', respectively, where $B=\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$

If $T:V \rightarrow W$ is a L.T. s.t.

$$[T(\mathbf{v}_1)]_{B'}=\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix},\ [T(\mathbf{v}_2)]_{B'}=\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots,\ [T(\mathbf{v}_n)]_{B'}=\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

then the $m\times n$ matrix whose n columns correspond to $[T(\mathbf{v}_i)]_{B'}$

$$A = \left[T(e_1) \mid T(e_2) \mid \cdots \mid T(e_n) \right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $\left[T(\mathbf{v}) \right]_{B'} = A[\mathbf{v}]_B$ for every $\mathbf{v}$ in V.

- Ex 5: (Finding a matrix relative to nonstandard bases)

Let $T: \ R^2 \to R^2$ be a L.T. defined by

$$T(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2)$$

Find the matrix of T relative to the basis

$B = \{(1, 2), (-1, 1)\}$ and $B' = \{(1, 0), (0, 1)\}$

Sol:

$$T(1, 2) = (3, 0) = 3(1, 0) + 0(0, 1)$$

$$T(-1, 1) = (0, -3) = 0(1, 0) - 3(0, 1)$$

$$\left[T(1, 2)\right]_{B'} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \ \left[T(-1, 1)\right]_{B'} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

the matrix for T relative to B and B'

$$A = \left[\left[T(1, 2)\right]_{B'} \quad \left[T(1, 2)\right]_{B'}\right] = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$$

- Ex 6:

For the L.T. $T: R^2 \to R^2$ given in Example 5, use the matrix A to find $T(\mathbf{v})$, where $\mathbf{v} = (2, 1)$

Sol:

$$\mathbf{v} = (2,1) = 1(1,2) - 1(-1,1)$$

$$B = \{(1, 2), (-1, 1)\}$$

$$\Rightarrow [\mathbf{v}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow [T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\Rightarrow T(\mathbf{v}) = 3(1,0) + 3(0,1) = (3,3)$$

$$B' = \{(1, 0), (0, 1)\}$$

- Check:

$$T(2,1) = (2+1, 2(2)-1) = (3,3)$$

- Notes:

(1) In the special case where $V = W$ and $B = B'$,

the matrix A is alled the matrix of T relative to the basis B

(2) $T : V \rightarrow V$: the identity t ransformat ion

$B = \{v_1, v_2, \cdots, v_n\}$: a basis for V

$\Rightarrow$ the matrix of T relative to the basis B

$$A = \left[[T(v_1)]_B, [T(v_2)]_B, \cdots, [T(v_n)]_B\right] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_n$$

Key Learning in Section 6.3

- Find the standard matrix for a linear transformation.
- Find the standard matrix for the composition of linear transformations and find the inverse of an invertible linear transformation.
- Find the matrix for a linear transformation relative to a nonstandard basis.

Keywords in Section 6.3

- standard matrix for T: T 的標準矩陣
- composition of linear transformations: 線性轉換的合成
- inverse linear transformation: 反線性轉換
- matrix of T relative to the bases B and B' : T對應於基底B到B'的矩陣
- matrix of T relative to the basis B: T對應於基底B的矩陣

6.4 Transition Matrices and Similarity

$T: V \to V$ (a L.T.)

$B = \{v_1, v_2, \cdots, v_n\}$ (a basis of V)

$B' = \{w_1, w_2, \cdots, w_n\}$ (a basis of V)

$A = \left[[T(v_1)]_B, [T(v_2)]_B, \cdots, [T(v_n)]_B \right]$ (matrix of T relative to B)

$A' = \left[[T(w_1)]_{B'}, [T(w_2)]_{B'}, \cdots, [T(w_n)]_{B'} \right]$ (matrix of T relative to B')

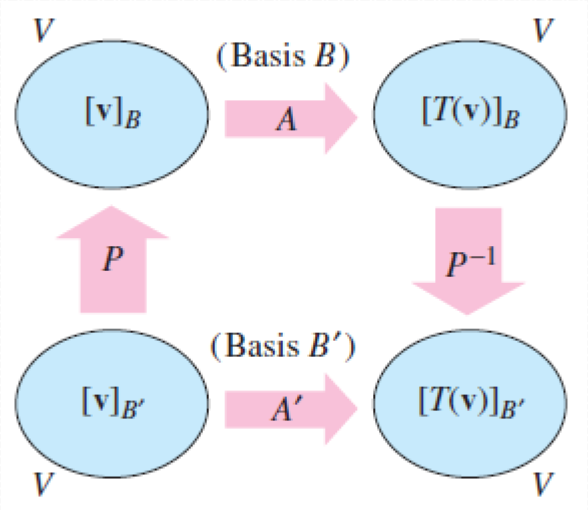
$P = \left[[w_1]_B, [w_2]_B, \cdots, [w_n]_B \right]$ (transition matrix from B' to B)

$P^{-1} = \left[[v_1]_{B'}, [v_2]_{B'}, \cdots, [v_n]_{B'} \right]$ (transition matrix from B to B')

$\therefore [\mathbf{v}]_B = P[\mathbf{v}]_{B'}, \ [\mathbf{v}]_{B'} = P^{-1}[\mathbf{v}]_B$

$[T(\mathbf{v})]_B = A[\mathbf{v}]_B$

$[T(\mathbf{v})]_{B'} = A'[\mathbf{v}]_{B'}$



- Two ways to get from $[\mathbf{v}]_{B'}$ to $[T(\mathbf{v})]_{B'}$:

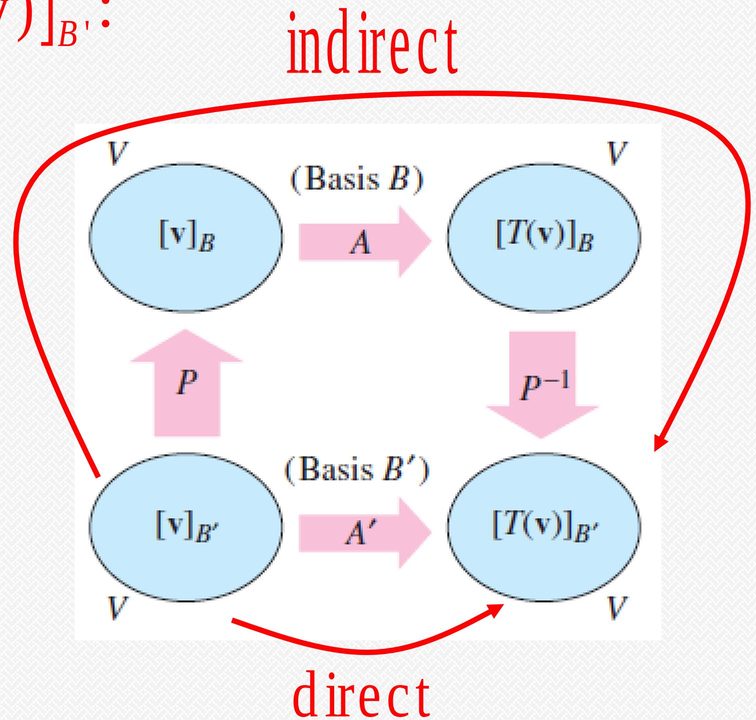
(1)(direct)

$$A'[\mathbf{v}]_{B'} = [T(\mathbf{v})]_{B'}$$

(2)(indirect)

$$P^{-1}AP[\mathbf{v}]_{B'} = [T(\mathbf{v})]_{B'}$$

$$\Rightarrow A' = P^{-1}AP$$



- Ex 1: (Finding a matrix for a linear transformation)

Find the matrix A' for $T: R^2 \rightarrow R^2$

$$T(x_1, x_2) = (2x_1 - 2x_2, -x_1 + 3x_2)$$

reletive to the basis $B' = \{(1, 0), (1, 1)\}$

Sol:

(I) $A' = \left[[T(1, 0)]_{B'} \quad [T(1, 1)]_{B'} \right]$

$$T(1, 0) = (2, -1) = 3(1, 0) - 1(1, 1) \Rightarrow [T(1, 0)]_{B'} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$T(1, 1) = (0, 2) = -2(1, 0) + 2(1, 1) \Rightarrow [T(1, 1)]_{B'} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\Rightarrow A' = \left[[T(1, 0)]_{B'} \quad [T(1, 1)]_{B'} \right] = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

(II) standard matrix for T (matrix of T relative to $B = \{(1,0), (0,1)\}$)

$$A = \begin{bmatrix} T(1,0) & T(0,1) \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$$

transition matrix from B' to B

$$P = \begin{bmatrix} [(1,0)]_B & [(1,1)]_B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

transition matrix from B to B'

$$P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

matrix of T relative B'

$$A' = P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

- Ex 2: (Finding a matrix for a linear transformation)

Let $B=\{(-3,2),(4,-2)\}$ and $B'=\{(-1,2),(2,-2)\}$ be basis for R^2,

and let $A=\begin{bmatrix}-2 & 7\\ -3 & 7\end{bmatrix}$ be the matrix for $T: R^2 \rightarrow R^2$ relative to B.

Find the matrix of T relative to B'.

Sol:

transition matrix from B' to B: $P=\left[\left[(-1,2)\right]_B \quad \left[(2,-2)\right]_B\right]=\begin{bmatrix}3 & -2\\ 2 & -1\end{bmatrix}$

transition matrix from B to B': $P^{-1}=\left[\left[(-3,2)\right]_{B'} \quad \left[(4,-2)\right]_{B'}\right]=\begin{bmatrix}-1 & 2\\ -2 & 3\end{bmatrix}$

matrix of T relative to B':

$$A'=P^{-1}AP=\begin{bmatrix}-1 & 2\\ -2 & 3\end{bmatrix}\begin{bmatrix}-2 & 7\\ -3 & 7\end{bmatrix}\begin{bmatrix}3 & -2\\ 2 & -1\end{bmatrix}=\begin{bmatrix}2 & 1\\ -1 & 3\end{bmatrix}$$

- Ex 3: (Finding a matrix for a linear transformation)

For the linear transformation $T: R^2 \to R^2$ given in Ex.2, find $[\mathbf{v}]_B$, $[T(\mathbf{v})]_B$ and $[T(\mathbf{v})]_{B'}$, for the vector $\mathbf{v}$ whose coordinate matrix is

$$[\mathbf{v}]_{B'} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

Sol:

$$[\mathbf{v}]_B = P[\mathbf{v}]_{B'} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ -5 \end{bmatrix}$$

$$[T(\mathbf{v})]_B = A[\mathbf{v}]_B = \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix}\begin{bmatrix} -7 \\ -5 \end{bmatrix} = \begin{bmatrix} -21 \\ -14 \end{bmatrix}$$

$$[T(\mathbf{v})]_{B'} = P^{-1}[T(\mathbf{v})]_B = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -21 \\ -14 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \end{bmatrix}$$

or $$[T(\mathbf{v})]_{B'} = A'[\mathbf{v}]_{B'} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \end{bmatrix}$$

- Similar matrix:

For square matrices A and A' of order n, A' is said to be similar to A if there exist an invertible matrix P *s.t.* $A' = P^{-1}AP$

- Thm 6.13: (Properties of similar matrices)

Let A, B, and C be square matrices of order n.

Then the following properties are true.

(1) A is similar to A.

(2) If A is similar to B, then B is similar to A.

(3) If A is similar to B and B is similar to C, then A is similar to C.

Pf:

$$(1)\ A = I_n A I_n$$

$$(2)\ A = P^{-1}BP \Rightarrow PAP^{-1} = P(P^{-1}BP)P^{-1}$$

$$PAP^{-1} = B \Rightarrow Q^{-1}AQ = B \ (Q = P^{-1})$$

- Ex 4: (Similar matrices)

$$(a)\ A=\begin{bmatrix}2 & -2\\ -1 & 3\end{bmatrix} \text{ and } A'=\begin{bmatrix}3 & -2\\ -1 & 2\end{bmatrix} \text{ are similar}$$

$$\text{because } A'=P^{-1}AP, \text{ where } P=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$$

$$(b)\ A=\begin{bmatrix}-2 & 7\\ -3 & 7\end{bmatrix} \text{ and } A'=\begin{bmatrix}2 & 1\\ -1 & 3\end{bmatrix} \text{ are similar}$$

$$\text{because } A'=P^{-1}AP, \text{ where } P=\begin{bmatrix}3 & -2\\ 2 & -1\end{bmatrix}$$

- Ex 5: (A comparison of two matrices for a linear transformation)

Suppose $A=\begin{bmatrix}1 & 3 & 0\\ 3 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}$ is the matrix for $T: R^3 \rightarrow R^3$ relative to the standard basis. Find the matrix for T relative to the basis

$$B' = \{(1,1,0),(1,-1,0),(0,0,1)\}$$

Sol:

The transition matrix from B' to the standard matrix

$$P = \big[[(1,1,0)]_B \quad [(1,-1,0)]_B \quad [(0,0,1)]_B\big] = \begin{bmatrix}1 & 1 & 0\\ 1 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow P^{-1} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & -\frac{1}{2} & 0\\ 0 & 0 & 1\end{bmatrix}$$

matrix of T relative to B':

$$A' = P^{-1}AP = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Notes: Computational advantages of diagonal matrices:

$$(1)\ D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix} \qquad D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

$$(2)\ D^T = D$$

$$(3)\ D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}, \quad d_i \neq 0$$

Key Learning in Section 6.4

- Find and use a matrix for a linear transformation.
- Show that two matrices are similar and use the properties of similar matrices.

Keywords in Section 6.4

- matrix of T relative to B: T 相對於B的矩陣
- matrix of T relative to B' : T 相對於B'的矩陣
- transition matrix from B' to B : 從B'到B的轉移矩陣
- transition matrix from B to B' : 從B到B'的轉移矩陣
- similar matrix: 相似矩陣

6.5 Applications of Linear Transformations

- Elementary matrices for linear transformations in R^2:

Reflection in y-axis

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reflection in x-axis

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Reflection in line $y = x$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Horizontal expansion ($k > 1$) or contraction ($0 < k < 1$)

$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

Vertical expansion ($k > 1$) or contraction ($0 < k < 1$)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

Horizontal shear

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Vertical shear

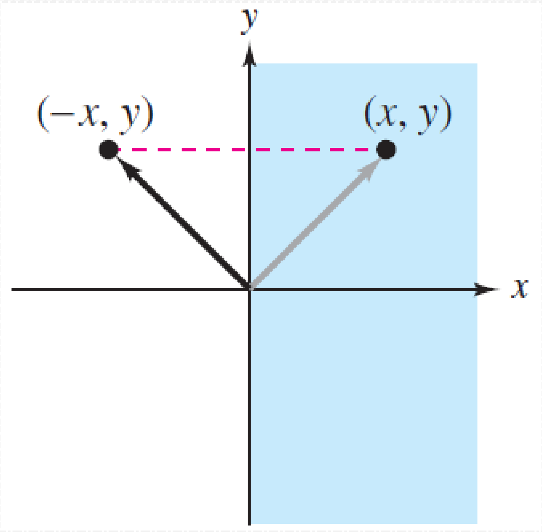
$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

- Ex 1: (Reflections in R^2)

a. Reflection in the y-axis

$$T(x,y)=(-x,y)$$

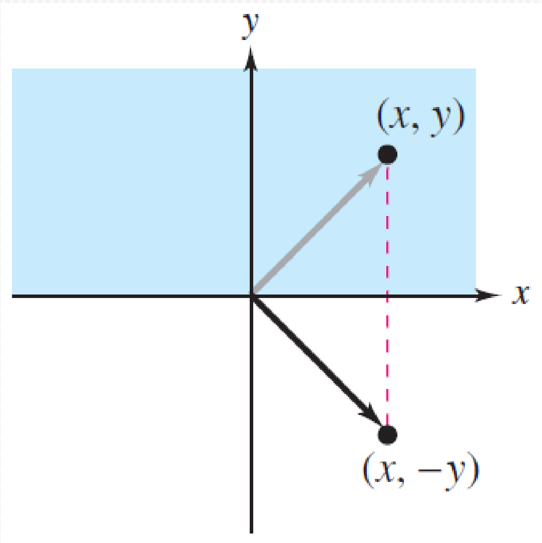
$$\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-x\\ y\end{bmatrix}$$



b. Reflection in the x-axis

$$T(x,y)=(x,-y)$$

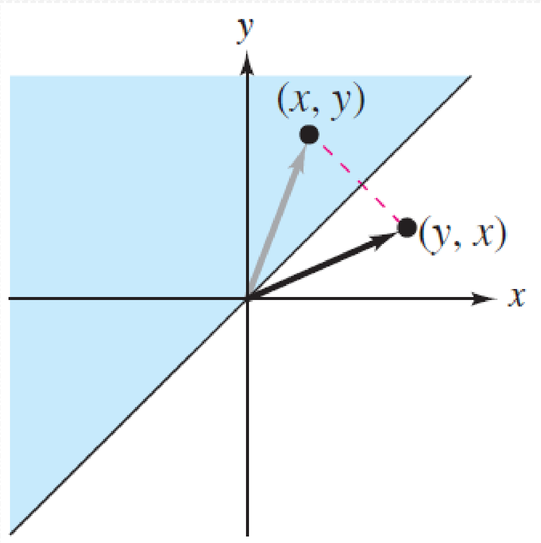
$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}x\\ -y\end{bmatrix}$$



c. Reflection in the line $y = x$

$$T(x,y)=(y,x)$$

$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}y\\ x\end{bmatrix}$$

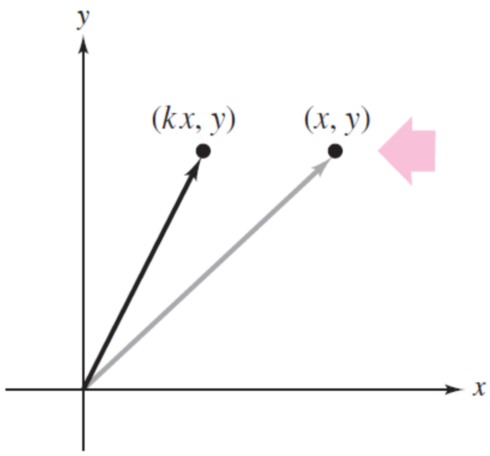


- Ex 2: (Expansions and Contractions in R^2)

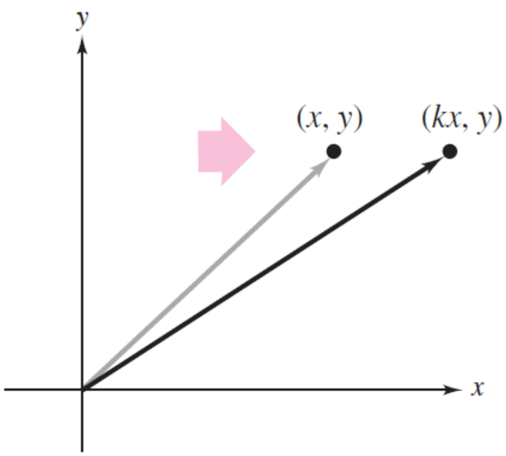
a. Horizontal contractions and expansions

$$T(x,y)=(kx,y)$$

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} kx \\ y \end{bmatrix}$$



Contraction $(0 < k < 1)$

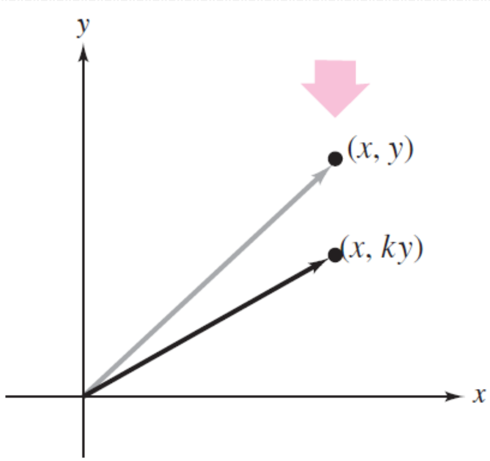


Expansion $(k > 1)$

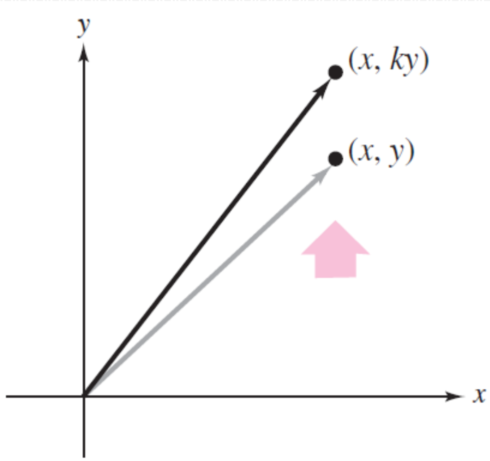
b. Vertical contractions and expansions

$$T(x,y)=(x,ky)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} x \\ ky \end{bmatrix}$$



Contraction $(0 < k < 1)$



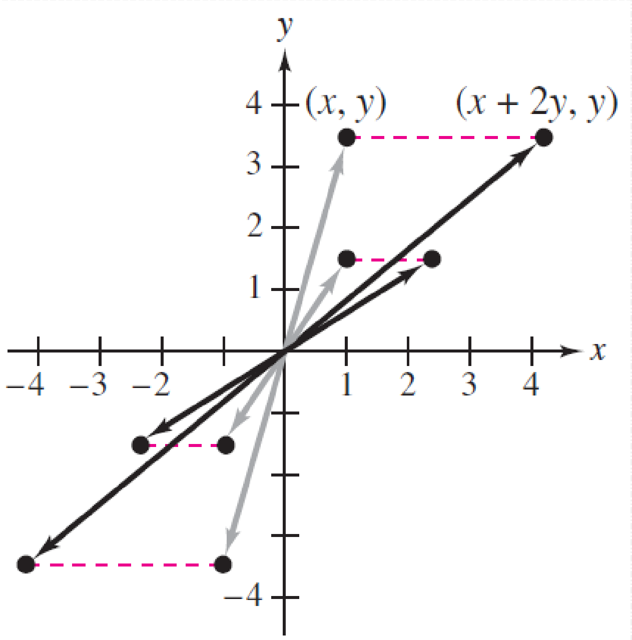
Expansion $(k > 1)$

- Ex 3: (Shears in R^2)

a. Horizontal shear

$$T(x, y) = (x + ky, y)$$

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ky \\ y \end{bmatrix}$$



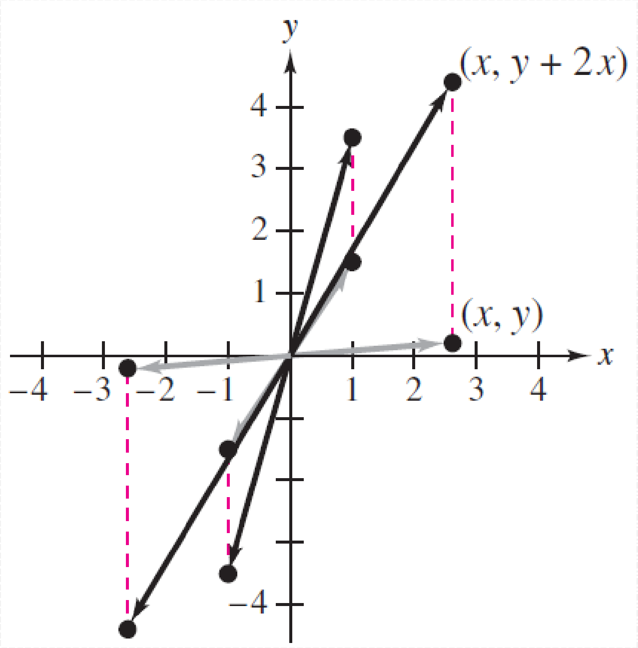
$$T(x, y) = (x + 2y, y)$$

Under this transformation, points in the upper half-plane "shear" to the right by amounts proportional to their y-coordinates. Points in the lower half-plane "shear" to the left by amounts proportional to the absolute values of their y-coordinates. Points on the x-axis do not move by this transformation.

b. Vertical shear

$$T(x, y) = (x, y + kx)$$

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ kx + y \end{bmatrix}$$



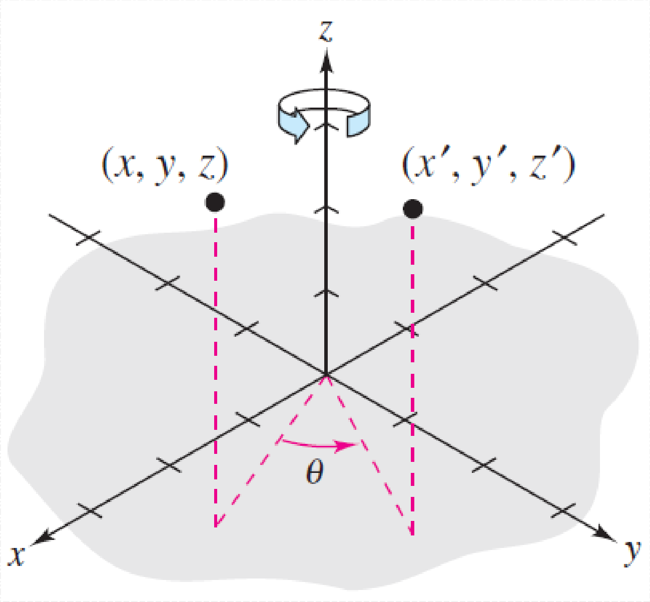
$$T(x, y) = (x, y + 2x)$$

Here, points in the right half-plane "shear" upward by amounts proportional to their x-coordinates. Points in the left half-plane "shear" downward by amounts proportional to the absolute values of their x-coordinates. Points on the y-axis do not move.

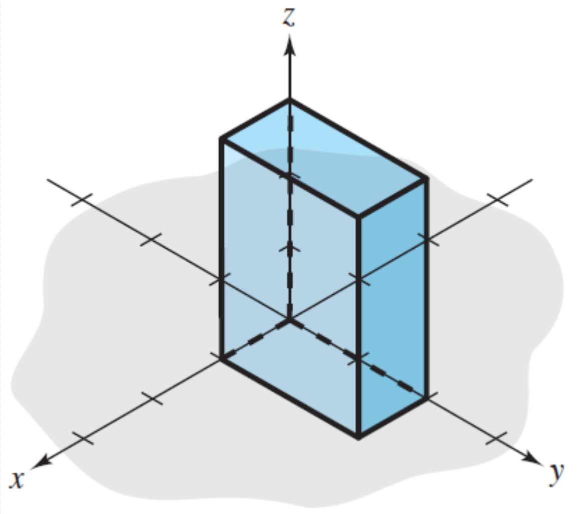
- Rotation in R^3:

Suppose you want to rotate the point (x, y, z) counterclockwise about the z-axis through an angle θ.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \end{bmatrix}$$



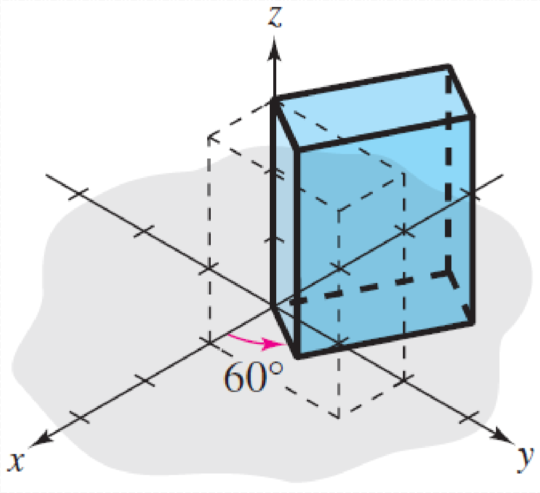
- Ex 4: (Rotation About the z-axis)



$$V_1(0,0,0) \quad V_2(1,0,0)$$
$$V_3(1,2,0) \quad V_4(0,2,0)$$
$$V_5(0,0,3) \quad V_6(1,0,3)$$
$$V_7(1,2,3) \quad V_8(0,2,3)$$

Sol:

a. A rotation of $60°$

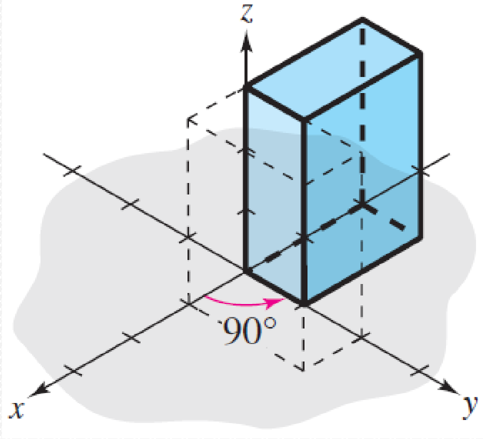


$$A=\begin{bmatrix}\cos 60° & -\sin 60° & 0\\ \sin 60° & \cos 60° & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1/2 & -\sqrt{3}/2 & 0\\ \sqrt{3}/2 & 1/2 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$V_1'(0,0,0) \quad V_2'(0.5,0.87,0) \quad V_3'(-1.23,1.87,0) \quad V_4'(-1.73,1,0)$$
$$V_5'(0,0,3) \quad V_6'(0.5,0.87,3) \quad V_7'(-1.23,1.87,3) \quad V_8'(-1.73,1,3)$$

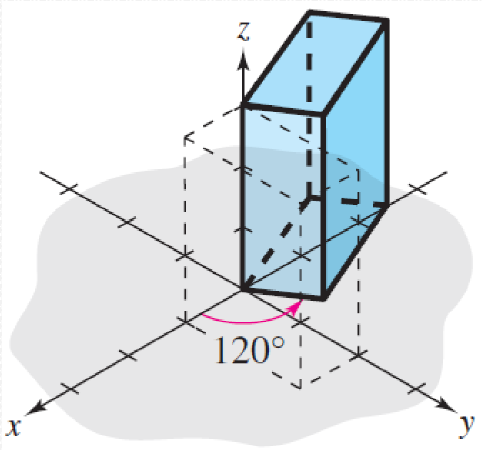
b. A rotation of $90°$

$$A=\begin{bmatrix}\cos 90° & -\sin 90° & 0\\ \sin 90° & \cos 90° & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$



c. A rotation of $120°$

$$A=\begin{bmatrix}\cos 120° & -\sin 120° & 0\\ \sin 120° & \cos 120° & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-1/2 & -\sqrt{3}/2 & 0\\ \sqrt{3}/2 & -1/2 & 0\\ 0 & 0 & 1\end{bmatrix}$$



Rotation about the *x*-axis

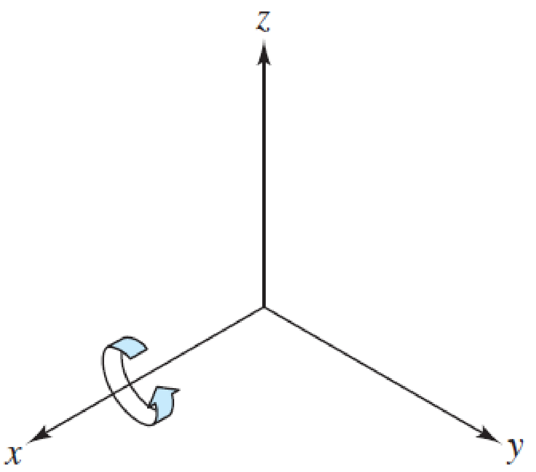
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Rotation about the *y*-axis

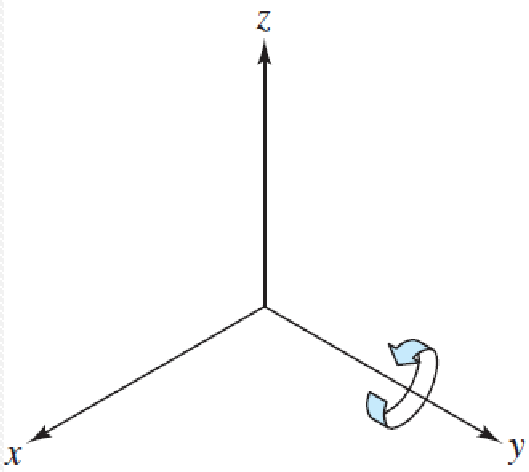
$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Rotation about the *z*-axis

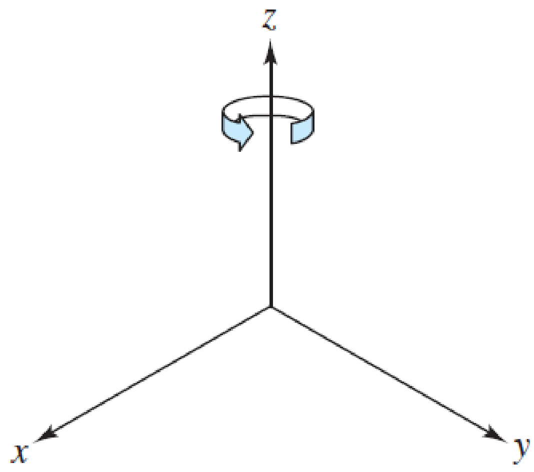
$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Rotation about *x*-axis



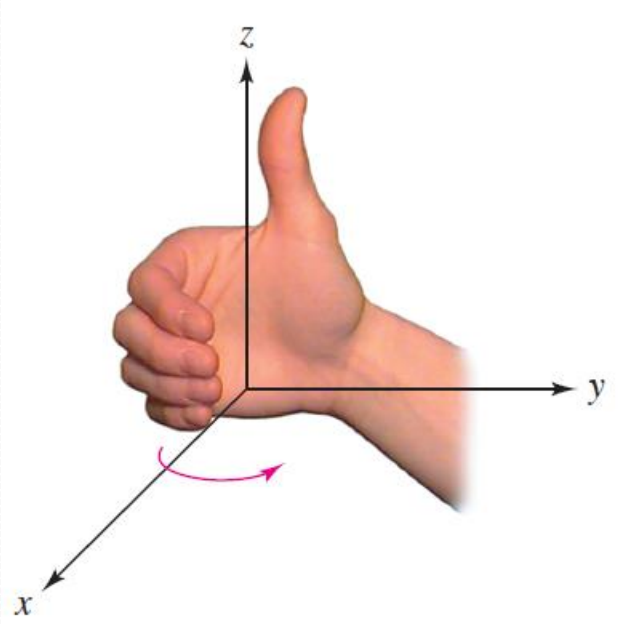
Rotation about *y*-axis



Rotation about *z*-axis

- Note:

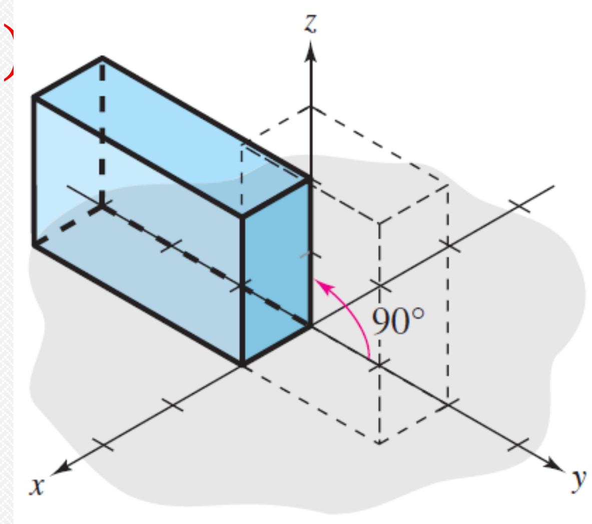
To illustrate the right-hand rule, imagine the thumb of your right hand pointing in the positive direction of an axis. The cupped fingers will point in the direction of counterclockwise rotation. The figure below shows counterclockwise rotation about the z-axis.



- Ex 5: (Rotation About the *x*-Axis and *y*-Axis)

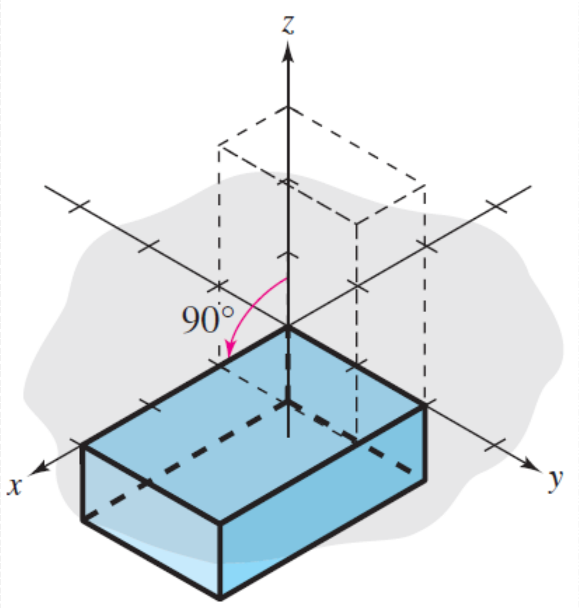
(a) A rotation of $90°$ about the x-axis

$$A=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos 90° & -\sin 90°\\ 0 & \sin 90° & \cos 90°\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}$$



(b) A rotation of $90°$ about the y-axis

$$A=\begin{bmatrix}\cos 90° & 0 & \sin 90°\\ 0 & 1 & 0\\ -\sin 90° & 0 & \cos 90°\end{bmatrix}=\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ -1 & 0 & 0\end{bmatrix}$$



Key Learning in Section 6.5

- Identify linear transformations defined by reflections, expansions, contractions, or shears in R^2.
- Use a linear transformation to rotate a figure in R^3.

Keywords in Section 6.5

- reflection: 鏡射
- expansion: 擴展
- contraction: 縮減
- share: 切變
- rectangular prism: 長方體

- Multivariate Statistics

Many multivariate statistical methods can use linear transformations. For instance, in a *multiple regression analysis*, there are two or more independent variables and a single dependent variable. A linear transformation is useful for finding weights to be assigned to the independent variables to predict the value of the dependent variable. Also, in a *canonical correlation analysis*, there are two or more independent variables and two or more dependent variables. Linear transformations can help find a linear combination of the independent variables to predict the value of a linear combination of the dependent variables.

- Control Systems

A control system, such as the one shown for a dairy factory, processes an input signal $\mathbf{x}_k$ and produces an output signal $\mathbf{x}_{k+1}$. Without external feedback, the **difference equation** $\mathbf{x}_{k+1} = A\mathbf{x}_k$, a linear transformation where $\mathbf{x}_i$ is an $n \times 1$ vector and A is an $n \times n$ matrix, can model the relationship between the input and output signals. Typically, however, a control system has external feedback, so the relationship becomes $\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$, where B is an $n \times m$ matrix and $\mathbf{u}_k$ is an $m \times 1$ input, or control, vector. A system is called *controllable* when it can reach any desired final state from its initial state in or fewer steps. If A and B make up a controllable system, then the rank of the *controllability matrix*

$$[B \quad AB \quad A^2B \;\; . \;\; . \;\; . \;\; A^{n-1}B]$$

is equal to n.

Circuit Design

Ladder networks are useful tools for electrical engineers involved in circuit design. In a ladder network, the output voltage *V* and current *I* of one circuit are the input voltage and current of the circuit next to it. In the ladder network shown below, linear transformations can relate the input and output of an individual circuit (enclosed in a dashed box). Using Kirchhoff's Voltage and Current Laws and Ohm's Law,

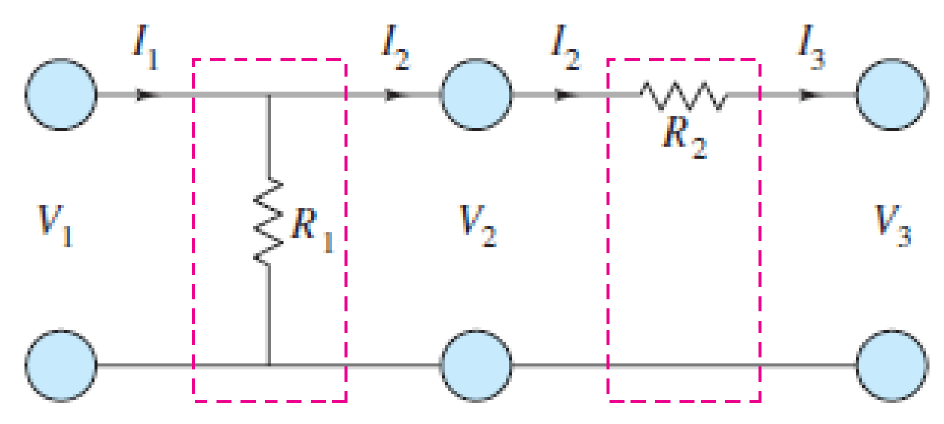
$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/R_1 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix}$$

and

$$\begin{bmatrix} V_3 \\ I_3 \end{bmatrix} = \begin{bmatrix} 1 & -1/R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}.$$

- Circuit Design

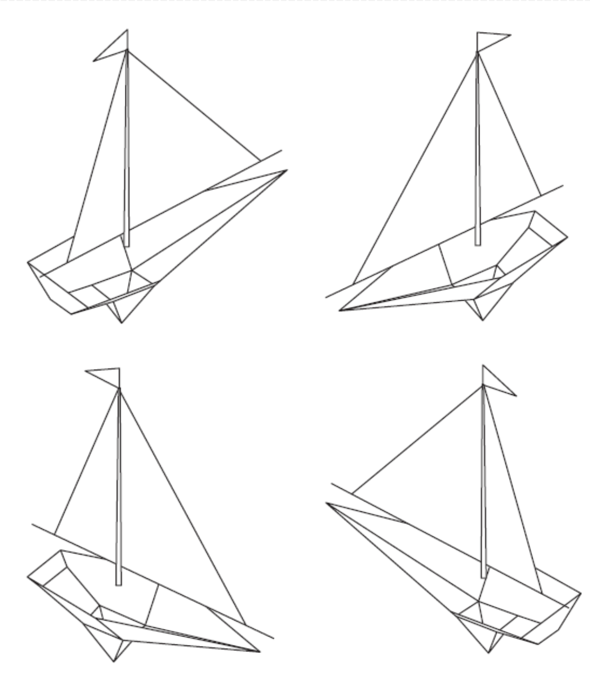
A composition can relate the input and output of the entire ladder network, that is, V_1 and I_1 to V_3 and I_3. Discussion on the composition of linear transformations begins on the following page.



- Weather

A **Leslie matrix,** named after British mathematician Patrick H. Leslie (1900–1974), can be used to find the age and growth distributions of a population over time. The entries in the first row of an $n \times n$ Leslie matrix L are the average numbers of offspring per member for each of n age classes. The entries in subsequent rows are pi in row i +1, column i and 0 elsewhere, where pi is the probability that an ith age class member will survive to become an (i +1)th age class member. If $\mathbf{x}_j$ is the age distribution vector for the jth time period, then the age distribution vector for the (j +1)th time period can be found using the linear transformation $\mathbf{x}_{j+1} = L\mathbf{x}_j$. You will study population growth models using Leslie matrices in more detail in Section 7.4.

- Computer Graphics



The use of computer graphics is common in many fields. By using graphics software, a designer can "see" an object before it is physically created. Linear transformations can be useful in computer graphics. To illustrate with a simplified example, only 23 points in R^3 were used to generate images of the toy boat shown in the figure at the left. Most graphics software can use such minimal information to generate views of an image from any perspective, as well as color, shade, and render as appropriate. Linear transformations, specifically those that produce rotations in R^3 can represent the different views. The remainder of this section discusses rotation in R^3.