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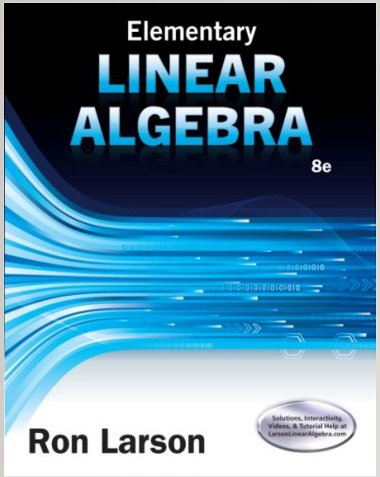

CHAPTER 7
EIGENVALUES AND EIGENVECTORS

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CH 7 Linear Algebra Applied

Diffusion (p.354)

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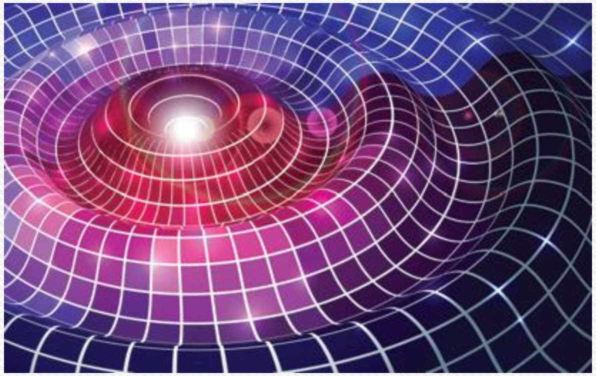

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7.1 Eigenvalues and Eigenvectors

- Eigenvalue problem:

  If A is an $n \times n$ matrix, do there exist nonzero vectors $\boldsymbol{x}$ in R^n such that $A\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{x}$?

- Eigenvalue and eigenvector:

  A : an $n \times n$ matrix

  λ : a scalar

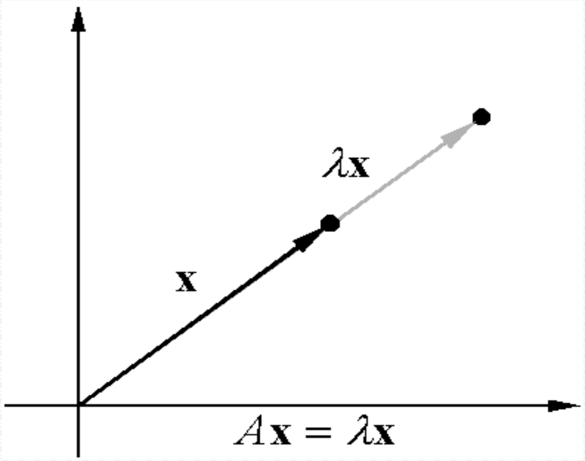
  $\boldsymbol{x}$: a nonzero vector in R^n

  Eigenvalue

  $$A\mathbf{x} = \lambda\mathbf{x}$$

  Eigenvector

- Geometrical Interpretation



- Ex 1: (Verifying eigenvalues and eigenvectors)

$$A=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix} \quad x_1=\begin{bmatrix}1\\0\end{bmatrix} \quad x_2=\begin{bmatrix}0\\1\end{bmatrix}$$

Eigenvalue

$$Ax_1=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}=2\begin{bmatrix}1\\0\end{bmatrix}=2x_1$$

Eigenvector

Eigenvalue

$$Ax_2=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix}=-1\begin{bmatrix}0\\1\end{bmatrix}=(-1)x_2$$

Eigenvector

- Thm 7.1: (The eigenspace of A corresponding to λ)

  If A is an $n \times n$ matrix with an eigenvalue λ, then the set of all eigenvectors of λ together with the zero vector is a subspace of R^n. This subspace is called **the eigenspace of λ** .

Pf:

x_1 and x_2 are eigenvectors corresponding to λ

$(i.e.\ \ Ax_1 = \lambda x_1,\ \ Ax_2 = \lambda x_2)$

$$(1)\ A(x_1 + x_2) = Ax_1 + Ax_2 = \lambda x_1 + \lambda x_2 = \lambda(x_1 + x_2)$$

$(i.e.\ \ x_1 + x_2$ is an eigenvector corresponding to $\lambda)$

$$(2)\ A(cx_1) = c(Ax_1) = c(\lambda x_1) = \lambda(cx_1)$$

$(i.e.\ \ cx_1$ is an eigenvector corresponding to $\lambda)$

■ Ex 3: (An example of eigenspaces in the plane)

Find the eigenvalues and corresponding eigenspaces of

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Sol:

If $\quad \mathrm{v} = (x, y)$

$$A\mathrm{v} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

For a vector on the *x*-axis

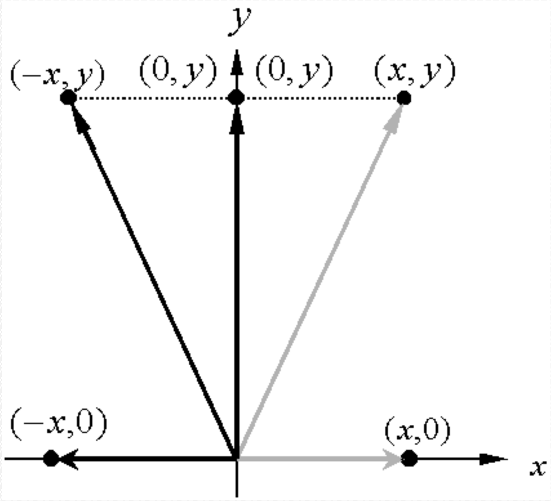
Eigenvalue $\lambda_1 = -1$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -x \\ 0 \end{bmatrix} = -1 \begin{bmatrix} x \\ 0 \end{bmatrix}$$

For a vector on the *y*-axis

Eigenvalue $\lambda_2 = 1$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = 1 \begin{bmatrix} 0 \\ y \end{bmatrix}$$



Geometrically, multiplying a vector (x, y) in R^2 by the matrix *A* corresponds to a reflection in the *y*-axis.

The eigenspace corresponding to $\lambda_1 = -1$ is the *x*-axis.

The eigenspace corresponding to $\lambda_2 = 1$ is the *y*-axis.

- Thm 7.2: (Finding eigenvalues and eigenvectors of a matrix $A \in M_{n \times n}$)

  Let A is an $n \times n$ matrix.

  (1) An eigenvalue of A is a scalar λ such that $\det(\lambda I - A) = 0$.

  (2) The eigenvectors of A corresponding to λ are the nonzero solutions of $(\lambda I - A)x = 0$.

- Note:

  $Ax = \lambda x \;\Rightarrow\; (\lambda I - A)x = 0$ (homogeneous system)

  If $(\lambda I - A)x = 0$ has nonzero solutions iff $\det(\lambda I - A) = 0$.

- Characteristic polynomial of $A \in M_{n \times n}$:

$$\det(\lambda I - A) = \left|(\lambda I - A)\right| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

- Characteristic equation of A:

$$\det(\lambda I - A) = 0$$

- Ex 4: (Finding eigenvalues and eigenvectors)

$$A=\begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

Sol: Characteristic equation:

$$\det(\lambda \mathrm{I}-A)=\begin{vmatrix} \lambda-2 & 12 \\ -1 & \lambda+5 \end{vmatrix}$$

$$=\lambda^2+3\lambda+2=(\lambda+1)(\lambda+2)=0$$

$$\Rightarrow \lambda=-1,-2$$

$$\text{Eigenvalue s}:\ \lambda_1=-1,\lambda_2=-2$$

$$(1)\lambda_1 = -1 \Rightarrow (\lambda_1 \mathrm{I} - A)x = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\because \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \ t \neq 0$$

$$(2)\lambda_2 = -2 \Rightarrow (\lambda_2 \mathrm{I} - A)x = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\because \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \ t \neq 0$$

$$\text{Check}: \ Ax = \lambda_i x$$

- Ex 5: (Finding eigenvalues and eigenvectors)

  Find the eigenvalues and corresponding eigenvectors for the matrix A. What is the dimension of the eigenspace of each eigenvalue?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0$$

Eigenvalue: $\lambda = 2$

The eigenspace of A corresponding to $\lambda = 2$:

$$(\lambda \mathrm{I} - A)x = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\because \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \neq 0$$

$$\left\{ s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \middle| s, t \in R \right\} : \text{the eigenspace of A corresponding to } \lambda = 2$$

Thus, the dimension of its eigenspace is 2.

- Notes:

  (1) If an eigenvalue λ_1 occurs as a multiple root (*k times*) for the characteristic polynominal, then λ_1 has multiplicity *k*.

  (2) The multiplicity of an eigenvalue is greater than or equal to the dimension of its eigenspace.

- Ex 6： Find the eigenvalues of the matrix A and find a basis for each of the corresponding eigenspaces.

$$A=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 5 & -10\\ 1 & 0 & 2 & 0\\ 1 & 0 & 0 & 3\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda \mathrm{I}-A|=\begin{vmatrix}\lambda-1 & 0 & 0 & 0\\ 0 & \lambda-1 & -5 & 10\\ -1 & 0 & \lambda-2 & 0\\ -1 & 0 & 0 & \lambda-3\end{vmatrix}$$

$$=(\lambda-1)^2(\lambda-2)(\lambda-3)=0$$

$$\text{Eigenvalues}: \lambda_1=1, \lambda_2=2, \lambda_3=3$$

$$(1)\ \lambda_1 = 1$$

$$\Rightarrow (\lambda_1 I - A)x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ s \\ 2t \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix},\ \ s,t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$ is a basis for the eigenspace of A corresponding to $\lambda = 1$

$(2)\lambda_2 = 2$

$$\Rightarrow (\lambda_2 \mathrm{I} - A)x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 10 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 10 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \ t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$$ is a basis for the eigenspace of A corresponding to $\lambda = 2$

$(3)\lambda_3 = 3$

$$\Rightarrow (\lambda_3 \mathrm{I} - A)x = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -5 & 10 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -5 & 10 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -5t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}, \ t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$$ is a basis for the eigenspace of A corresponding to $\lambda = 3$

- **Thm 7.3: (Eigenvalues of triangular matrices)**

  If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.

- **Ex 7: (Finding eigenvalues for diagonal and triangular matrices)**

$$(a)\ A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix} \qquad (b)\ A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Sol:

$$(a)\ |\lambda I - A| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 1 & \lambda - 1 & 0 \\ -5 & -3 & \lambda + 3 \end{vmatrix} = (\lambda - 2)(\lambda - 1)(\lambda + 3)$$

$$\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -3$$

$$(b)\ \lambda_1 = -1, \lambda_2 = 2, \lambda_3 = 0, \lambda_4 = -4, \lambda_5 = 3$$

- Eigenvalues and eigenvectors of linear transformations:

A number λ is called an eigenvalue of a linear transformation $T: V \rightarrow V$ if there is a nonzero vector $\mathbf{x}$ such that $T(\mathbf{x}) = \lambda \mathbf{x}$. The vector $\mathbf{x}$ is called an eigenvector of T corresponding to λ, and the set of all eigenvectors of λ (with the zero vector) is called the eigenspace of λ.

- Ex 8: (Finding eigenvalues and eigenspaces)

Find the eigenvalues and corresponding eigenspaces

$$A=\begin{bmatrix}1 & 3 & 0\\ 3 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}.$$

Sol:

$$|\lambda I - A| = \begin{vmatrix}\lambda-1 & -3 & 0\\ -3 & \lambda-1 & 0\\ 0 & 0 & \lambda+2\end{vmatrix} = (\lambda+2)^2(\lambda-4)$$

$$\text{eigenvalues}: \lambda_1 = 4,\ \lambda_2 = -2$$

The eigenspaces for these two eigenvalues are as follows.

$B_1 = \{(1, 1, 0)\}$ Basis for $\lambda_1 = 4$

$B_2 = \{(1, -1, 0), (0, 0, 1)\}$ Basis for $\lambda_2 = -2$

- Notes:

(1) Let $T: R^3 \rightarrow R^3$ be the linear transformation whose standard matrix is A in Ex. 8, and let B' be the basis of R^3 made up of three linear independent eigenvectors found in Ex. 8. Then A', the matrix of T relative to the basis B', is diagonal.

$$B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$$

Eigenvectors of A

$$A' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Eigenvalues of A

(2) The main diagonal entries of the matrix A' are the eigenvalues of A.

Key Learning in Section 7.1

- Verify eigenvalues and corresponding eigenvectors.
- Find eigenvalues and corresponding eigenspaces.
- Use the characteristic equation to find eigenvalues and eigenvectors, and find the eigenvalues and eigenvectors of a triangular matrix.
- Find the eigenvalues and eigenvectors of a linear transformation.

Keywords in Section 7.1

- eigenvalue problem: 特徵值問題
- eigenvalue: 特徵值
- eigenvector: 特徵向量
- characteristic polynomial: 特徵多項式
- characteristic equation: 特徵方程式
- eigenspace: 特徵空間
- multiplicity: 重根數

7.2 Diagonalization

- Diagonalization problem:

For a square matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is diagonal?

- Diagonalizable matrix:

A square matrix A is called **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP$ is **a diagonal matrix**.

(P diagonalizes A)

- Notes:

(1) If there exists an invertible matrix P such that $B = P^{-1}AP$, then two square matrices A and B are called **similar**.

(2) The eigenvalue problem is related closely to the diagonalization problem.

- Thm 7.4: (Similar matrices have the same eigenvalues)

If A and B are similar $n \times n$ matrices, then they have the same eigenvalues.

Pf:

$$A \text{ and } B \text{ are similar} \Rightarrow B = P^{-1}AP$$

$$\begin{aligned} |\lambda I - B| &= |\lambda I - P^{-1}AP| = |P^{-1}\lambda I P - P^{-1}AP| = |P^{-1}(\lambda I - A)P| \\ &= |P^{-1}||\lambda I - A||P| = |P^{-1}||P||\lambda I - A| = |P^{-1}P||\lambda I - A| \\ &= |\lambda I - A| \end{aligned}$$

A and B have the same characteristic polynomial.

Thus A and B have the same eigenvalues.

- Ex 1: (A diagonalizable matrix)

$$A=\begin{bmatrix}1 & 3 & 0\\ 3 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I-A|=\begin{vmatrix}\lambda-1 & -3 & 0\\ -3 & \lambda-1 & 0\\ 0 & 0 & \lambda+2\end{vmatrix}=(\lambda-4)(\lambda+2)^2=0$$

$$\text{Eigenvalues}: \lambda_1=4, \lambda_2=-2, \lambda_3=-2$$

$$(1)\lambda=4\Rightarrow \text{Eigenvector}: p_1=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

$$(2)\lambda = -2 \Rightarrow \text{Eigenvector} \ : p_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \ p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P = [p_1 \quad p_2 \quad p_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Notes:

$$(1)\ P = [p_2 \quad p_1 \quad p_3] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$(2)\ P = [p_2 \quad p_3 \quad p_1] = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

- Thm 7.5: (Condition for diagonalization)

An $n \times n$ matrix A is diagonalizable if and only if

it has n linearly independent eigenvectors.

Pf:

$(\Rightarrow) A$ is diagonalizable

there exists an invertible P s.t. $D = P^{-1}AP$ is diagonal

Let $P = [\mathbf{p}_1 \mid \mathbf{p}_2 \mid \cdots \mid \mathbf{p}_n]$ and $D = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$

$$PD = [\mathbf{p}_1 \mid \mathbf{p}_2 \mid \cdots \mid \mathbf{p}_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$= [\lambda_1\mathbf{p}_1 \mid \lambda_2\mathbf{p}_2 \mid \cdots \mid \lambda_n\mathbf{p}_n]$$

$$AP = A[\mathbf{p}_1 \mid \mathbf{p}_2 \mid \cdots \mid \mathbf{p}_n] = [A\mathbf{p}_1 \mid A\mathbf{p}_2 \mid \cdots \mid A\mathbf{p}_n]$$

$\because AP = PD$

$\therefore A\mathbf{p}_i = \lambda_i \mathbf{p}_i, \; i = 1, 2, \ldots, n$

($i.e.$ the column vector $\mathbf{p}_i$ of P are eigenvectors of A)

$\because P$ is invertible $\Rightarrow \mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$ are linearly independent.

$\therefore$ A has n linearly independent eigenvectors.

$(\Leftarrow)$ A has n linearly independent eigenvectors $p_1, p_2, \cdots p_n$

with corresponding eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_n$

i.e. $A\mathbf{p}_i = \lambda_i \mathbf{p}_i, \; i = 1, 2, \ldots, n$

Let $P = [\mathbf{p}_1 \mid \mathbf{p}_2 \mid \cdots \mid \mathbf{p}_n]$

$$
\begin{aligned}
AP &= A[\mathbf{p}_1 \mid \mathbf{p}_2 \mid \cdots \mid \mathbf{p}_n] \\
&= [A\mathbf{p}_1 \mid A\mathbf{p}_2 \mid \cdots \mid A\mathbf{p}_n] \\
&= [\lambda_1\mathbf{p}_1 \mid \lambda_2\mathbf{p}_2 \mid \cdots \mid \lambda_n\mathbf{p}_n] \\
&= [\mathbf{p}_1 \mid \mathbf{p}_2 \mid \cdots \mid \mathbf{p}_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = PD
\end{aligned}
$$

$\because \mathbf{p}_1, \mathbf{p}_1, \cdots, \mathbf{p}_n$ are linearly independen t $\Rightarrow P$ is invertible

$\therefore P^{-1}AP = D$

$\Rightarrow A$ is diagonaliz able

Note: If n linearly independent vectors do not exist, then an $n \times n$ matrix A is not diagonalizable.

- Ex 4: (A matrix that is not diagonalizable)

Show that the following matrix is not diagonalizable.

$$A=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda \mathrm{I}-A|=\begin{vmatrix}\lambda-1 & -2\\ 0 & \lambda-1\end{vmatrix}=(\lambda-1)^2=0$$

Eigenvalue : $\lambda_1=1$

$$\lambda \mathrm{I}-A=I-A=\begin{bmatrix}0 & -2\\ 0 & 0\end{bmatrix}\sim\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\Rightarrow \text{Eigenvecto r}:\ p_1=\begin{bmatrix}1\\ 0\end{bmatrix}$$

A does not have two (n=2) linearly independent eigenvectors, so A is not diagonalizable.

- Steps for diagonalizing an $n \times n$ square matrix:

Step 1: Find n linearly independent eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$

for A with corresponding eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$

Step 2: Let $P = [\mathbf{p}_1 \mid \mathbf{p}_2 \mid \cdots \mid \mathbf{p}_n]$

Step 3:

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \text{ where } A\mathbf{p}_i = \lambda_i \mathbf{p}_i, \; i = 1, 2, \ldots, n$$

Note:

The order of the eigenvalues used to form P will determine the order in which the eigenvalues appear on the main diagonal of D.

- Ex 5: (Diagonalizing a matrix)

$$A=\begin{bmatrix}1 & -1 & -1\\ 1 & 3 & 1\\ -3 & 1 & -1\end{bmatrix}$$

Find a matrix P such that $P^{-1}AP$ is diagonal.

Sol: Characteristic equation:

$$|\lambda \mathrm{I}-A|=\begin{vmatrix}\lambda-1 & 1 & 1\\ -1 & \lambda-3 & -1\\ 3 & -1 & \lambda+1\end{vmatrix}=(\lambda-2)(\lambda+2)(\lambda-3)=0$$

Eigenvalue s: $\lambda_1=2, \lambda_2=-2, \lambda_3=3$

$\lambda_1 = 2$

$$\Rightarrow \lambda_1 \mathrm{I} - A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \text{Eigenvecto r}: \ p_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda_2 = -2$

$$\Rightarrow \lambda_2 \mathrm{I} - A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t \\ -\frac{1}{4}t \\ t \end{bmatrix} = \frac{1}{4}t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \Rightarrow \text{Eigenvecto r}: \ p_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$\lambda_3 = 3$

$$\Rightarrow \lambda_3 \mathrm{I} - A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \text{Eigenvecto r}: \; p_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{Let } \; P = [\, p_1 \quad p_2 \quad p_3 \,] = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix}$$

$$\Rightarrow \quad P^{-1} A P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- Notes: k is a positive integer

(1) $D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$

(2) $D = P^{-1}AP$

$$\begin{aligned} \Rightarrow D^k &= (P^{-1}AP)^k \\ &= (P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP) \\ &= P^{-1}A(PP^{-1})A(PP^{-1})\cdots(PP^{-1})AP \\ &= P^{-1}AA\cdots AP \\ &= P^{-1}A^kP \end{aligned}$$

$$\therefore A^k = PD^kP^{-1}$$

- Thm 7.6: (Sufficient conditions for diagonalization)

If an $n \times n$ matrix A has n distinct eigenvalues, then the corresponding eigenvectors are linearly independent and A is diagonalizable.

- Ex 7: (Determining whether a matrix is diagonalizable)

$$A=\begin{bmatrix}1 & -2 & 1\\ 0 & 0 & 1\\ 0 & 0 & -3\end{bmatrix}$$

Sol: Because A is a triangular matrix,

its eigenvalues are the main diagonal entries.

$$\lambda_1=1, \lambda_2=0, \lambda_3=-3$$

These three values are distinct, so A is diagonalizable. (Thm.7.6)

- Ex 8: (Finding a diagonalizing matrix for a linear transformation)

Let $T: R^3 \to R^3$ be the linear transformation given by

$$T(x_1, x_2, x_3) = (x_1 - x_2 - x_3,\ x_1 + 3x_2 + x_3,\ -3x_1 + x_2 - x_3)$$

Find a basis B for R^3 such that the matrix for T relative to B is diagonal.

Sol: The standard matrix for T is given by

$$A = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

From Ex. 5, there are three distinct eigenvalues

$$\lambda_1 = 2, \lambda_2 = -2, \lambda_3 = 3$$

so A is diagonalizable. (Thm. 7.6)

Thus, the three linearly independent eigenvectors found in Ex. 5

$$p_1 = (-1, 0, 1),\ p_2 = (1, -1, 4),\ p_3 = (-1, 1, 1)$$

can be used to form the basis *B*. That is

$$B = \{p_1, p_2, p_3\} = \{(-1, 0, 1), (1, -1, 4), (-1, 1, 1)\}$$

The matrix for *T* relative to this basis is

$$\begin{aligned} D &= \left[[T(p_1)]_B \ \ [T(p_2)]_B \ \ [T(p_3)]_B \right] \\ &= \left[[Ap_1]_B \ \ [Ap_2]_B \ \ [Ap_3]_B \right] \\ &= \left[[\lambda_1 p_1]_B \ \ [\lambda_2 p_2]_B \ \ [\lambda_3 p_3]_B \right] \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \end{aligned}$$

Key Learning in Section 7.2

- Find the eigenvalues of similar matrices, determine whether a matrix A is diagonalizable, and find a matrix P such that $P^{-1}AP$ is diagonal.
- Find, for a linear transformation T: $V \to V$ a basis B for V such that the matrix T for B relative to is diagonal.

Keywords in Section 7.2

- diagonalization problem: 對角化問題
- diagonalization: 對角化
- diagonalizable matrix: 可對角化矩陣

7.3 Symmetric Matrices and Orthogonal Diagonalization

- Symmetric matrix:

A square matrix A is **symmetric** if it is equal to its transpose:

$$A = A^T$$

- Ex 1: (Symmetric matrices and nonsymetric matrices)

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix}$$ (symmetric)

$$B = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}$$ (symmetric)

$$C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -4 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$ (nonsymmetric)

- Thm 7.7: (Eigenvalues of symmetric matrices)

If A is an $n \times n$ symmetric matrix, then the following properties are true.

(1) A is diagonalizable.

(2) All eigenvalues of A are real.

(3) If λ is an eigenvalue of A with multiplicity k, then λ has k linearly independent eigenvectors. That is, the eigenspace of λ has dimension k.

- Ex 2:

Prove that a symmetric matrix is diagonalizable.

$$A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

Pf: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - a & -c \\ -c & \lambda - b \end{vmatrix} = \lambda^2 - (a+b)\lambda + ab - c^2 = 0$$

As a quadratic in λ, this polynomial has a discriminant of

$$\begin{aligned}(a+b)^2 - 4(ab - c^2) &= a^2 + 2ab + b^2 - 4ab + 4c^2 \\ &= a^2 - 2ab + b^2 + 4c^2 \\ &= \underline{(a-b)^2 + 4c^2} \ \geq 0\end{aligned}$$

(1) $(a-b)^2+4c^2=0$

$\Rightarrow\ a=b,\ c=0$

$A=\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ is a matrix of diagonal.

(2) $(a-b)^2+4c^2>0$

The characteristic polynomial of A has two distinct real roots, which implies that A has two distinct real eigenvalues. Thus, A is diagonalizable.

- Orthogonal matrix:

A square matrix P is called **orthogonal** if it is invertible and

$$P^{-1} = P^T$$

- Ex 4: (Orthogonal matrices)

(a) $P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is orthogonal because $P^{-1} = P^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(b) $P = \begin{bmatrix} \frac{3}{5} & 0 & \frac{-4}{5} \\ 0 & 1 & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}$ is orthogonal because $P^{-1} = P^T = \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ 0 & 1 & 0 \\ \frac{-4}{5} & 0 & \frac{3}{5} \end{bmatrix}$.

- Thm 7.8: (Properties of orthogonal matrices)

An $n \times n$ matrix P is orthogonal if and only if its column vectors form an orthogonal set.

- Ex 5: (An orthogonal matrix)

$$P=\begin{bmatrix}\frac{1}{3} & \frac{2}{3} & \frac{2}{3}\\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0\\ \frac{-2}{3\sqrt{5}} & \frac{-4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}}\end{bmatrix}$$

Sol: If P is a orthogonal matrix, then $P^{-1}=P^T \Rightarrow PP^T=\text{I}$

$$PP^T=\begin{bmatrix}\frac{1}{3} & \frac{2}{3} & \frac{2}{3}\\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0\\ \frac{-2}{3\sqrt{5}} & \frac{-4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}}\end{bmatrix}\begin{bmatrix}\frac{1}{3} & \frac{-2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}}\\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{-4}{3\sqrt{5}}\\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\text{I}$$

$$\text{Let } \mathbf{p}_1=\begin{bmatrix}\frac{1}{3}\\ \frac{-2}{\sqrt{5}}\\ \frac{-2}{3\sqrt{5}}\end{bmatrix}, \mathbf{p}_2=\begin{bmatrix}\frac{2}{3}\\ \frac{1}{\sqrt{5}}\\ \frac{-4}{3\sqrt{5}}\end{bmatrix}, \mathbf{p}_3=\begin{bmatrix}\frac{2}{3}\\ 0\\ \frac{5}{3\sqrt{5}}\end{bmatrix}$$

produces

$$\mathbf{p}_1 \cdot \mathbf{p}_2 = \mathbf{p}_1 \cdot \mathbf{p}_3 = \mathbf{p}_2 \cdot \mathbf{p}_3 = 0$$

$$\|\mathbf{p}_1\| = \|\mathbf{p}_2\| = \|\mathbf{p}_3\| = 1$$

$\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is an orthonormal set.

- Thm 7.9: (Properties of symmetric matrices)

Let A be an $n \times n$ symmetric matrix. If λ_1 and λ_2 are distinct eigenvalues of A, then their corresponding eigenvectors x_1 and x_2 are orthogonal.

- Ex 6: (Eigenvectors of a symmetric matrix)

Show that any two eigenvecto rs of $A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$

correspond ing to distinct eigenvalue s are orthogonal .

- Sol: Characteristic function

$$|\lambda \mathrm{I}-A|=\begin{vmatrix}\lambda-3 & -1\\ -1 & \lambda-3\end{vmatrix}=\lambda^2-6\lambda+8=(\lambda-2)(\lambda-4)=0$$

$\Rightarrow$ Eigenvalue s: $\lambda_1=2, \lambda_2=4$

$$(1)\ \lambda_1=2\Rightarrow \lambda_1\mathrm{I}-A=\begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}\sim\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\Rightarrow \mathrm{x}_1=s\begin{bmatrix}-1\\ 1\end{bmatrix},\ s\neq 0$$

$$(2)\ \lambda_2=4\Rightarrow \lambda_2\mathrm{I}-A=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\sim\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}\Rightarrow \mathrm{x}_2=t\begin{bmatrix}1\\ 1\end{bmatrix},\ t\neq 0$$

$$\mathrm{x}_1\cdot\mathrm{x}_2=\begin{bmatrix}-s\\ s\end{bmatrix}\cdot\begin{bmatrix}t\\ t\end{bmatrix}=st-st=0\ \Rightarrow\ \mathrm{x}_1 \text{ and } \mathrm{x}_2 \text{ are orthogonal .}$$

- Thm 7.10: (Fundamental theorem of symmetric matrices)

  Let A be an $n \times n$ matrix. Then A is orthogonally diagonalizable and has real eigenvalue if and only if A is symmetric.

- Orthogonal diagonalization of a symmetric matrix:

  Let A be an $n \times n$ symmetric matrix.

  (1) Find all eigenvalues of A and determine the multiplicity of each.

  (2) For each eigenvalue of multiplicity 1, choose a unit eigenvector.

  (3) For each eigenvalue of multiplicity $k \geq 2$, find a set of k linearly independent eigenvectors. If this set is not orthonormal, apply Gram-Schmidt orthonormalization process.

  (4) The composite of steps 2 and 3 produces an orthonormal set of n eigenvectors. Use these eigenvectors to form the columns of P. The matrix $P^{-1}AP = P^{T}AP = D$ will be diagonal.

- Ex 7: (Determining whether a matrix is orthogonally diagonalizable)

	Symmetric matrix	Orthogonally diagonalizable
$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	○	○
$A_2 = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 8 \\ -1 & 8 & 0 \end{bmatrix}$	×	×
$A_3 = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$	×	×
$A_4 = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$	○	○

- Ex 9: (Orthogonal diagonalization)

Find an orthogonal matrix P that diagonalizes A.

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{bmatrix}$$

Sol:

(1) $|\lambda I - A| = (\lambda - 3)^2 (\lambda + 6) = 0$

$$\lambda_1 = -6, \lambda_2 = 3 \text{ (has a multiplicity of 2)}$$

(2) $\lambda_1 = -6, \quad v_1 = (1, -2, 2) \Rightarrow u_1 = \dfrac{v_1}{\|v_1\|} = (\tfrac{1}{3}, \tfrac{-2}{3}, \tfrac{2}{3})$

(3) $\lambda_2 = 3, \quad v_2 = (2, 1, 0), \ v_3 = (-2, 0, 1)$

Linear Independent

Gram-Schmidt Process:

$$w_2 = v_2 = (2, 1, 0), \quad w_3 = v_3 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2 = (\tfrac{-2}{5}, \tfrac{4}{5}, 1)$$

$$u_2 = \frac{w_2}{\|w_2\|} = (\tfrac{2}{\sqrt{5}}, \tfrac{1}{\sqrt{5}}, 0), \quad u_3 = \frac{w_3}{\|w_3\|} = (\tfrac{-2}{3\sqrt{5}}, \tfrac{4}{3\sqrt{5}}, \tfrac{5}{3\sqrt{5}})$$

$$(4)\ P = [p_1 \ p_2 \ p_3] = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}} \\ \frac{-2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix}$$

$$\Rightarrow \quad P^{-1}AP = P^{T}AP = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Key Learning in Section 7.3

- Recognize, and apply properties of, symmetric matrices.
- Recognize, and apply properties of, orthogonal matrices.
- Find an orthogonal matrix P that orthogonally diagonalizes a symmetric matrix A.

Keywords in Section 7.3

- symmetric matrix: 對稱矩陣
- orthogonal matrix: 正交矩陣
- orthonormal set: 單範正交集
- orthogonal diagonalization: 正交對角化

7.4 Applications of Eigenvalues and Eigenvectors

- Population growth:

The **age distribution vector x** represents the number of population members in each age class, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \begin{matrix} \text{Number in first age class} \\ \text{Number in second age class} \\ \\ \text{Number in } n\text{th age class} \end{matrix}$$

Multiplying **the age transition matrix** by the age distribution vector for a specific time period produces the age distribution vector for the next time period. That is,

$$L\mathbf{x}_j = \mathbf{x}_{j+1} \qquad L = \begin{bmatrix} b_1 & b_2 & \cdots & b_{n-1} & b_n \\ p_1 & 0 & \cdots & 0 & 0 \\ 0 & p_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & p_{n-1} & 0 \end{bmatrix}$$

- Ex 1: (A Population Growth Model)

The current age distribution vector is

$$\mathbf{x}_1 = \begin{bmatrix} 24 \\ 24 \\ 20 \end{bmatrix} \begin{matrix} 0 \le \text{age} < 1 \\ 1 \le \text{age} < 2 \\ 2 \le \text{age} \le 3 \end{matrix}$$

and the age transition matrix is

$$L = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$$

After 1 year, the age distribution vector will be

$$\mathbf{x}_2 = L\mathbf{x}_1 = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 24 \\ 24 \\ 20 \end{bmatrix} = \begin{bmatrix} 304 \\ 12 \\ 12 \end{bmatrix} \begin{matrix} 0 \le \text{age} < 1 \\ 1 \le \text{age} < 2 \\ 2 \le \text{age} \le 3 \end{matrix}$$

- Ex 2: (Finding a Stable Age Distribution Vector)

To solve this problem, find an eigenvalue and a corresponding eigenvector **x** such that $L\mathbf{x} = \mathbf{x}$. The characteristic polynomial of L is

$$|\lambda I - L| = (\lambda+1)^2(\lambda-2)$$

(check this), which implies that the eigenvalues are −1 and 2. Choosing the positive value, let λ=2. Verify that the corresponding eigenvectors are of the form

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16t \\ 4t \\ t \end{bmatrix} = t\begin{bmatrix} 16 \\ 4 \\ 1 \end{bmatrix}$$

- Ex 2: (Finding a Stable Age Distribution Vector)

For example, if $t = 2$, then the initial age distribution vector is

$$\mathbf{x}_1 = \begin{bmatrix} 32 \\ 8 \\ 2 \end{bmatrix} \begin{matrix} 0 \le \text{age} < 1 \\ 1 \le \text{age} < 2 \\ 2 \le \text{age} \le 3 \end{matrix}$$

and the age distribution vector for the next year is

$$\mathbf{x}_2 = L\mathbf{x}_1 = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 32 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} 64 \\ 16 \\ 4 \end{bmatrix} \begin{matrix} 0 \le \text{age} < 1 \\ 1 \le \text{age} < 2 \\ 2 \le \text{age} \le 3 \end{matrix}$$

Notice that the ratio of the three age classes is still 16 : 4 : 1, and so the percent of the population in each age class remains the same.

- Systems of Linear Differential Equations (Calculus)

A system of first-order linear differential equations

$$
\begin{aligned}
y_1' &= a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n \\
y_2' &= a_{21}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n \\
\\
y_n' &= a_{n1}y_1 + a_{n2}y_2 + \ldots + a_{nn}y_n
\end{aligned}
$$

where each y_i is a function of t and $y_1' = \dfrac{dy_i}{d_t}$. If you let

$$
\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \text{and } A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}
$$

then the system can be written in matrix form as $\mathbf{y}' = A\mathbf{y}$.

- Ex 3: (Solving a System of Linear Differential Equations)

Solve the system of linear differential equations.

$$\begin{aligned} y_1' &= 4y_1 \\ y_2' &= -y_2 \\ y_3' &= 2y_3 \end{aligned}$$

Sol:

From calculus, you know that the solution of the differential equation $y' = ky$ is

$$y = Ce^{kt}$$

So, the solution of the system is

$$\begin{aligned} y_1 &= C_1 e^{4t} \\ y_2 &= C_2 e^{-t} \\ y_3 &= C_3 e^{2t} \end{aligned}$$

- Ex 4: (Solving a System of Linear Differential Equations)

Solve the system of linear differential equations.

$$y_1' = 3y_1 + 2y_2$$
$$y_2' = 6y_1 - y_2$$

first find a matrix P that diagonalizes $A = \begin{bmatrix} 3 & 2 \\ 6 & -1 \end{bmatrix}$.

Verify that the eigenvalues of A are 1 = −3 and 2 = 5, and that the corresponding eigenvectors are $\mathbf{p}_1 = [1 \quad -3]^T$ and $\mathbf{p}_2 = [1 \quad 1]^T$. Diagonalize A using the matrix P whose columns consist of $\mathbf{p}_1$ and $\mathbf{p}_2$ to obtain

$$P = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}, P^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}, \text{and } P^{-1}AP = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$$

- Quadratic Forms

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

Quadratic equation

$$a'(x')^2 + c'(y')^2 + d'x' + e'y' + f' = 0$$

$$A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$$

$$ax^2 + bxy + cy^2$$

Quadratic form

- Ex 5: (Finding the Matrix of the Quadratic Form)

(a) $4x^2 + 9y^2 - 36 = 0$

(b) $13x^2 - 10xy + 13y^2 - 72 = 0$

Sol:

(a) $a = 4$, $b = 0$, and $c = 9$, so the matrix is

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

Diagonal matrix (no *xy*-term)

(b) $a = 13$, $b = -10$, and $c = 13$, so the matrix is

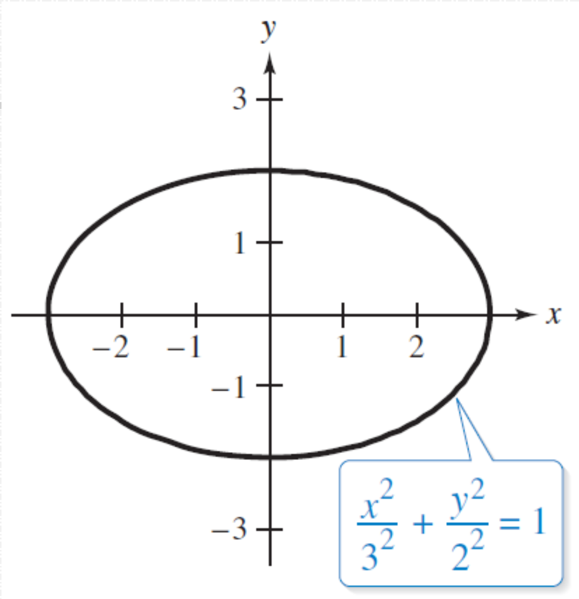
$$A = \begin{bmatrix} 13 & -5 \\ -5 & 13 \end{bmatrix}$$

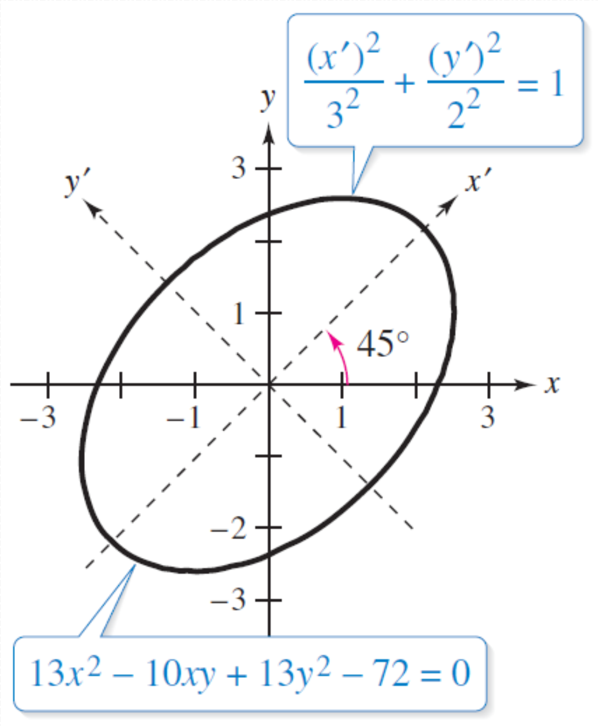
Nondiagonal matrix (*xy*-term)

$$\frac{x^2}{3^2}+\frac{y^2}{2^2}=1$$

$$\frac{(x')^2}{3^2}+\frac{(y')^2}{2^2}=1$$

$$X=\begin{bmatrix} x \\ y \end{bmatrix}$$





$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$X^T AX + [d \quad e]X + f = [x \quad y]\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + [d \quad e]\begin{bmatrix} x \\ y \end{bmatrix} + f$$

$$= ax^2 + bxy + cy^2 + dx + ey + f$$

$$P^T X = X' = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$X = PX'$$

$$X^T AX = (PX')^T A(PX') = (X')^T P^T APX' = (X')^T DX'$$

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad |P| = 1$$

- Principal Axes Theorem

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

$$X = PX' \qquad |P| = 1$$

$$P^T AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\lambda_1 (x')^2 + \lambda_2 (y')^2 + \begin{bmatrix} d & e \end{bmatrix} PX' + f = 0$$

- Ex 6: (Rotation of a Conic)

$$13x^2 - 10xy + 13y^2 - 72 = 0$$

Sol:

$$A = \begin{bmatrix} 13 & -5 \\ -5 & 13 \end{bmatrix}$$

eigenvalue

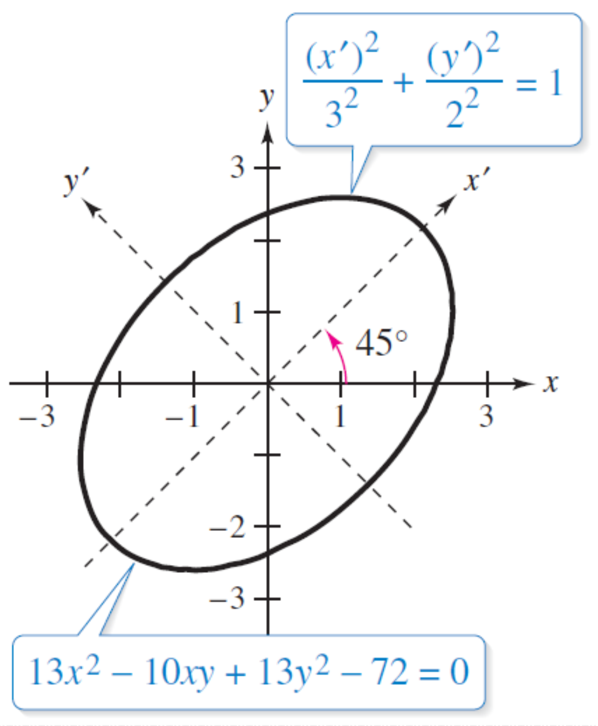
$$\lambda_1 = 8 \text{ and } \lambda_2 = 18$$

$$8(x')^2 + 18(y')^2 - 72 = 0$$

$$\frac{(x')^2}{3^2} + \frac{(y')^2}{2^2} = 1$$

eigenvector

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



Orthogonal matrix

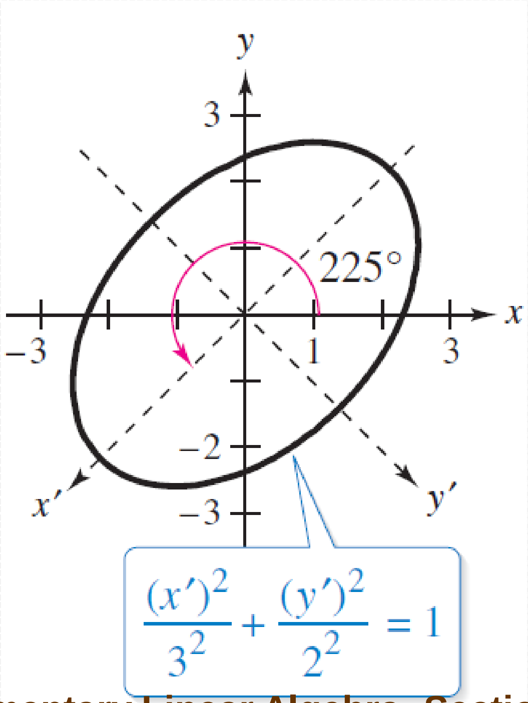
$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \end{array}$$

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\lambda_1 = 8, \lambda_2 = 18$$

$$\theta = 225°$$

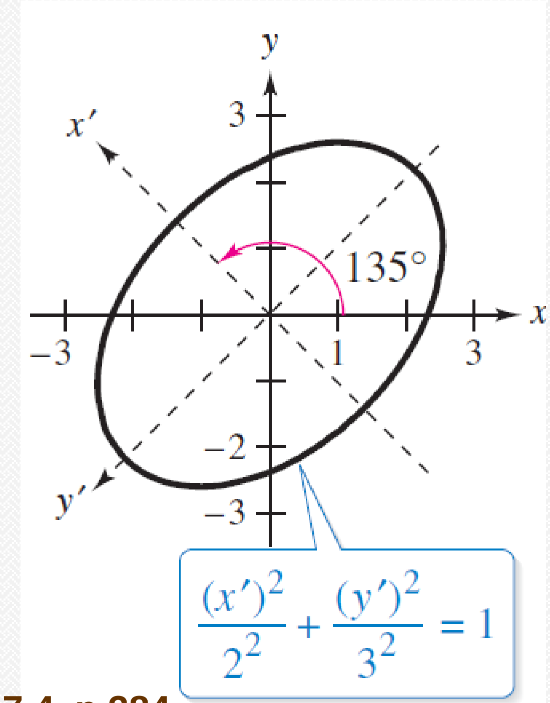


$$\begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \end{array}$$

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\lambda_1 = 18, \lambda_2 = 8$$

$$\theta = 135°$$

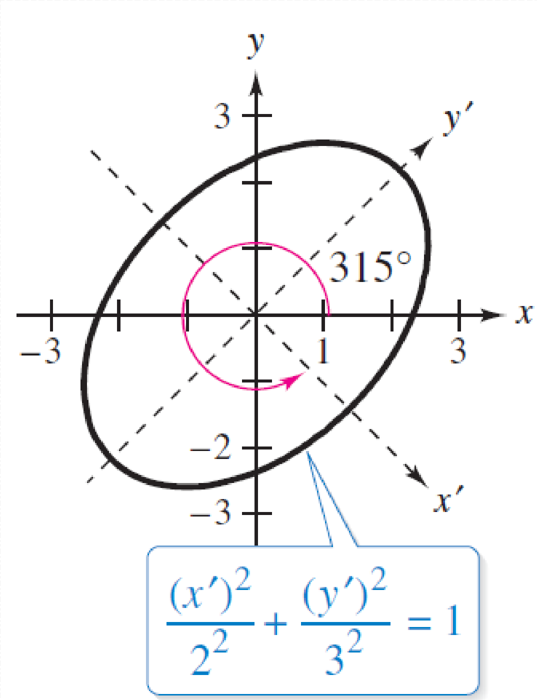


$$\begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \end{array}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\lambda_1 = 18, \lambda_2 = 8$$

$$\theta = 315°$$



- Ex 7: (Rotation of a Conic)

$$3x^2 - 10xy + 3y^2 + 16\sqrt{2}x - 32 = 0$$

Sol:

$$A = \begin{bmatrix} 3 & -5 \\ -5 & 3 \end{bmatrix}$$

eigenvalue

$$\lambda_1 = 8 \text{ and } \lambda_2 = -2$$

eigenvector

$$x_1 = (-1, 1) \text{ and } x_2 = (-1, -1)$$

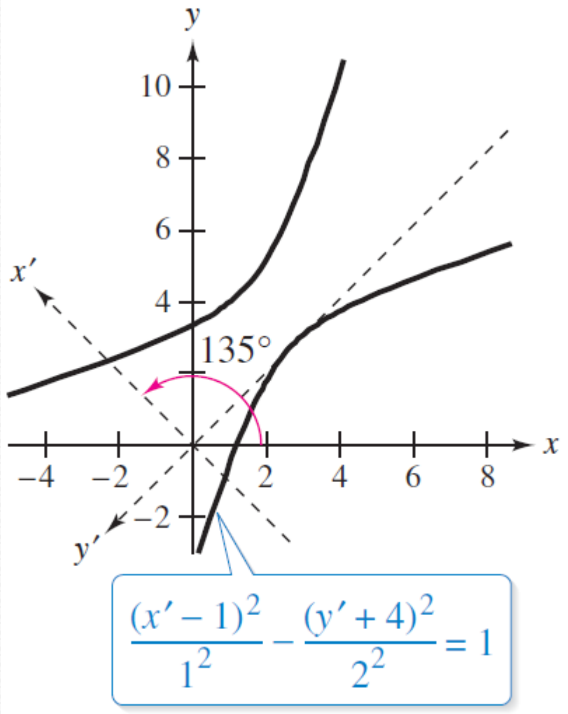
orthogonal matrix

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad |P| = 1$$

$$[d \quad e]PX' = [16\sqrt{2} \quad 0]\begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = -16x' - 16y'$$

$$8(x')^2 - 2(y')^2 - 16x' - 16y' - 32 = 0$$

$$\frac{(x'-1)^2}{1^2} - \frac{(y'+4)^2}{2^2} = 1$$



$$ax^2+by^2+cz^2+dxy+exz+fyz+gx+hy+iz+j=0$$ Quadratic equation

$$A=\begin{bmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{bmatrix}$$

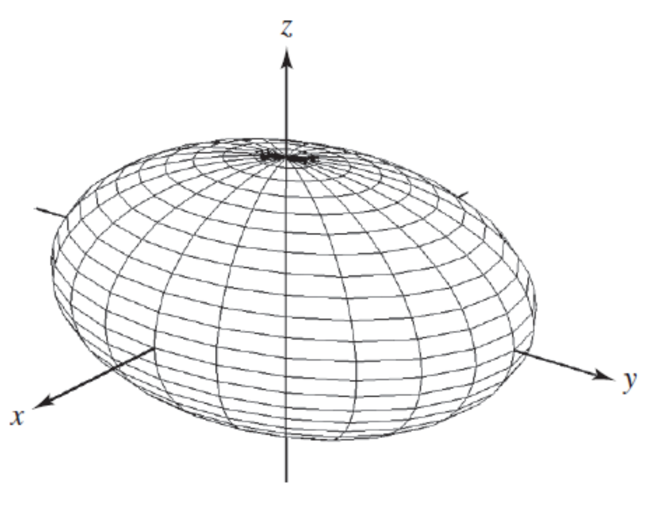
$$ax^2+by^2+cz^2+dxy+exz+fyz$$ Quadratic form

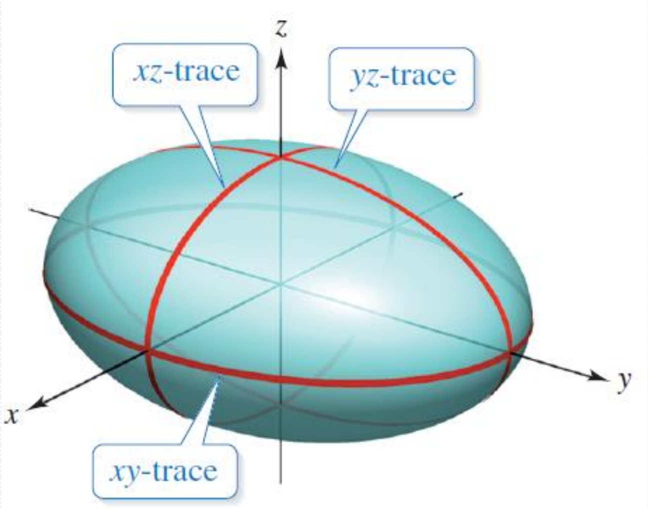
- Ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Trace	*Plane*
Ellipse	Parallel to *xy*-plane
Ellipse	Parallel to *xz*-plane
Ellipse	Parallel to *yz*-plane

The surface is a sphere when $a = b = c \neq 0$.



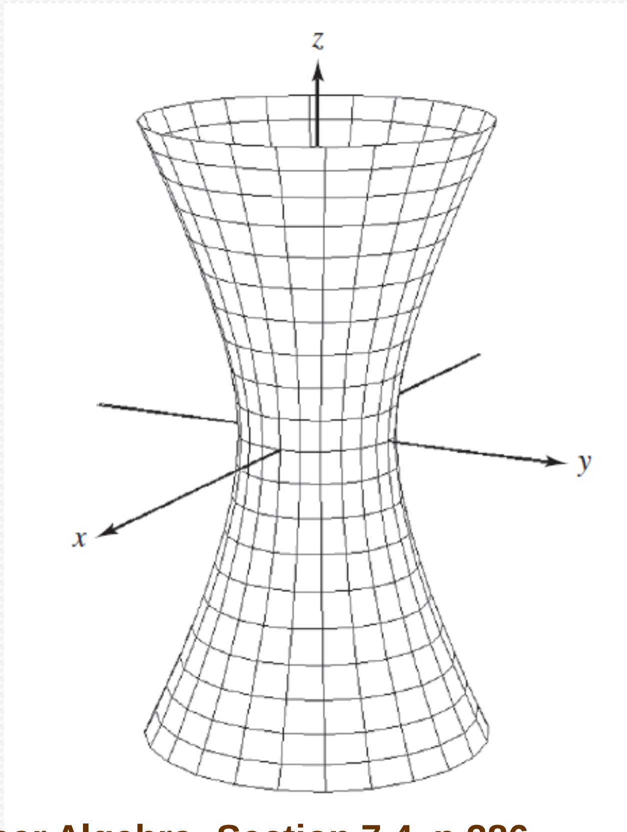


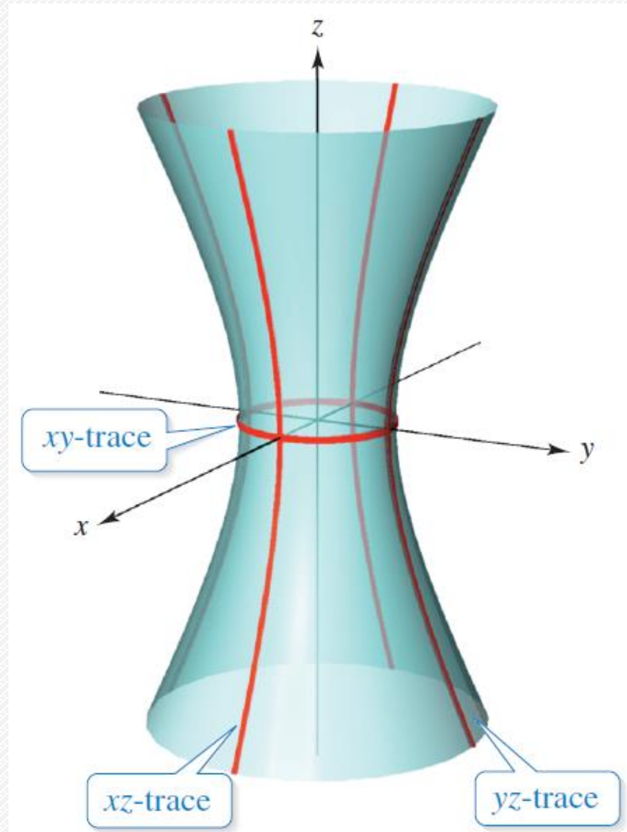
- **Hyperboloid of One Sheet**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is negative.



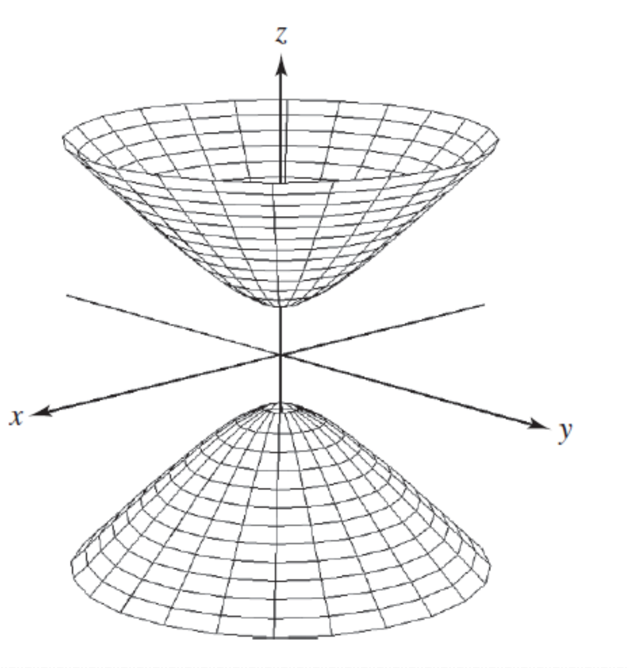


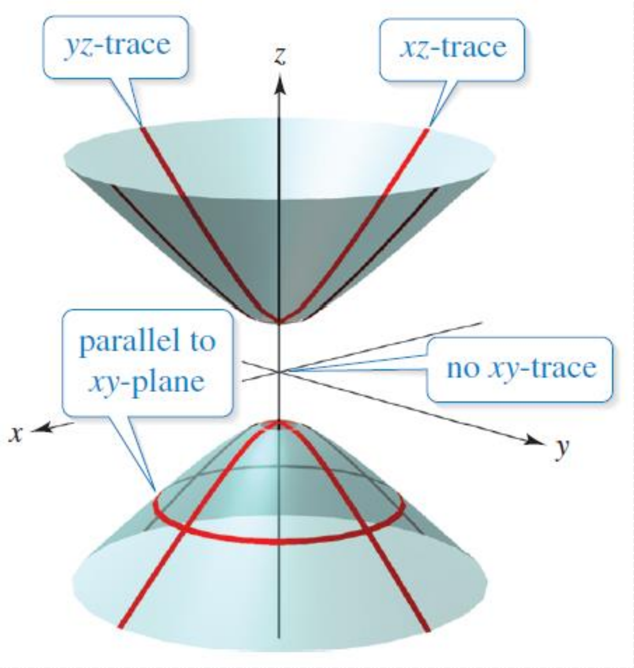
- Hyperboloid of Two Sheet

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Trace	*Plane*
Ellipse	Parallel to *xy*-plane
Hyperbola	Parallel to *xz*-plane
Hyperbola	Parallel to *yz*-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is positive. There is no trace in the coordinate plane perpendicular to this axis.



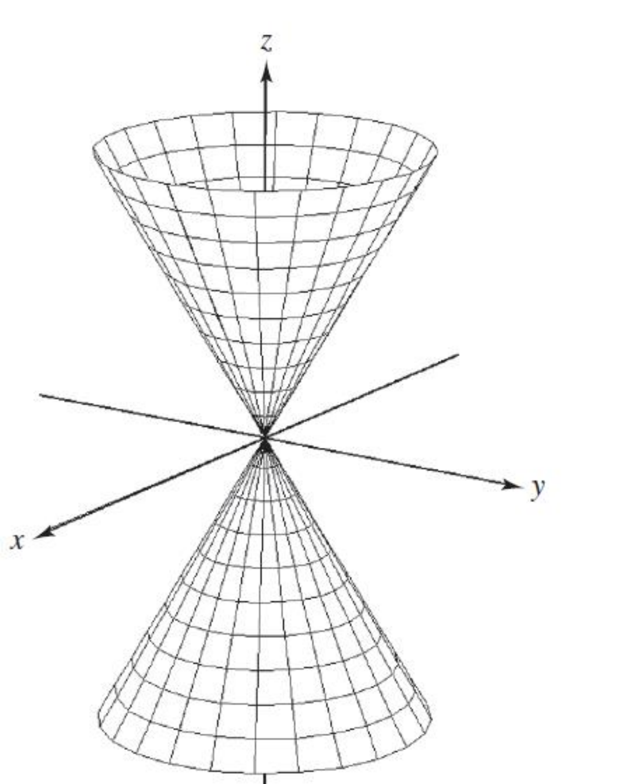


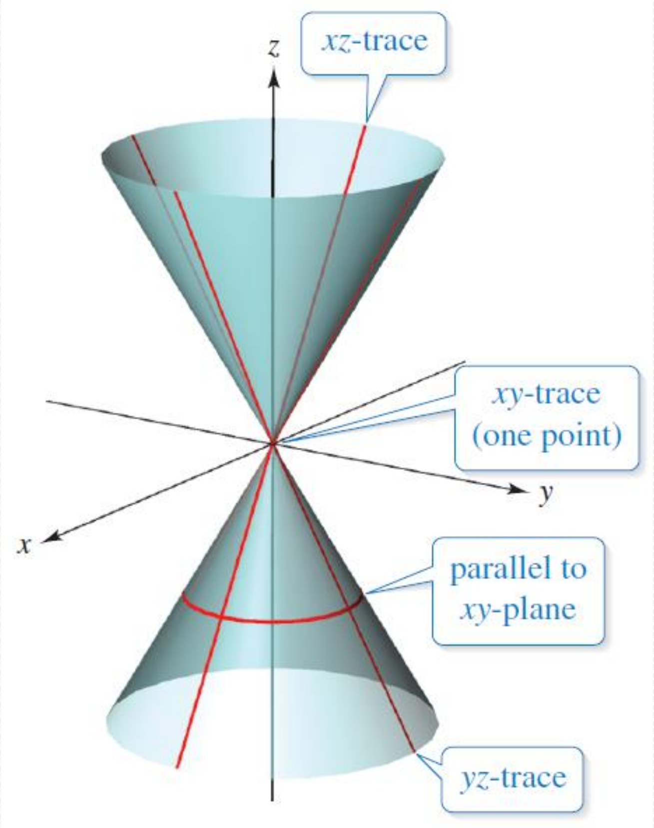
- Elliptic Cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

Trace	Plane
Ellipse	Parallel to *xy*-plane
Hyperbola	Parallel to *xz*-plane
Hyperbola	Parallel to *yz*-plane

The axis of the cone corresponds to the variable whose coefficient is negative. The traces in the coordinate planes parallel to this axis are intersecting lines.



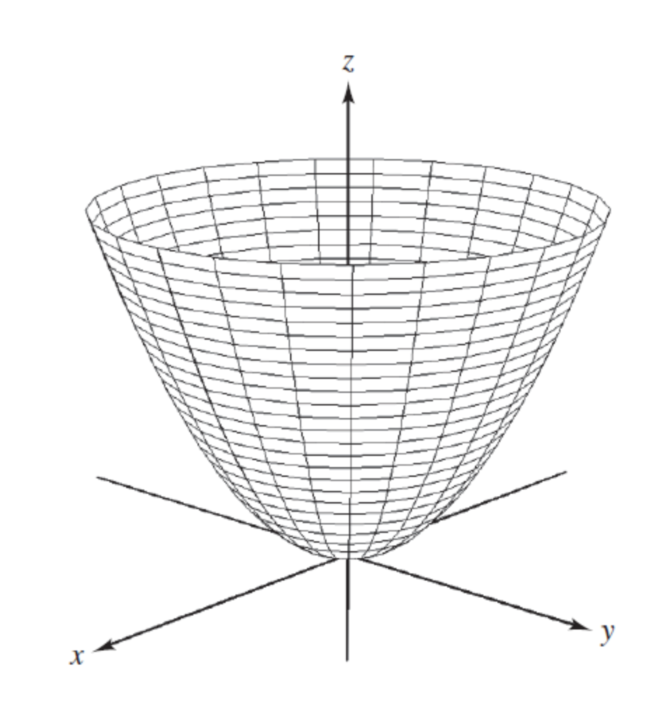


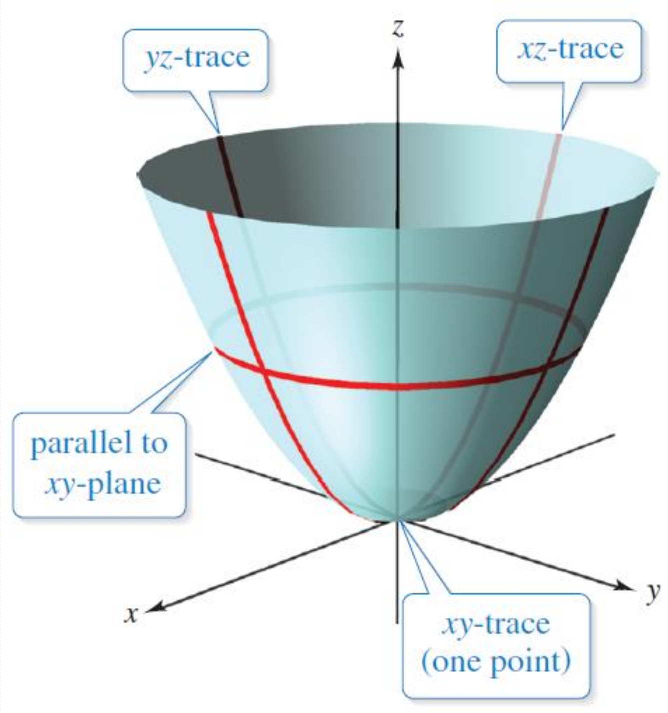
- Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Trace	*Plane*
Ellipse	Parallel to *xy*-plane
Parabola	Parallel to *xz*-plane
Parabola	Parallel to *yz*-plane

The axis of the paraboloid corresponds to the variable raised to the first power.



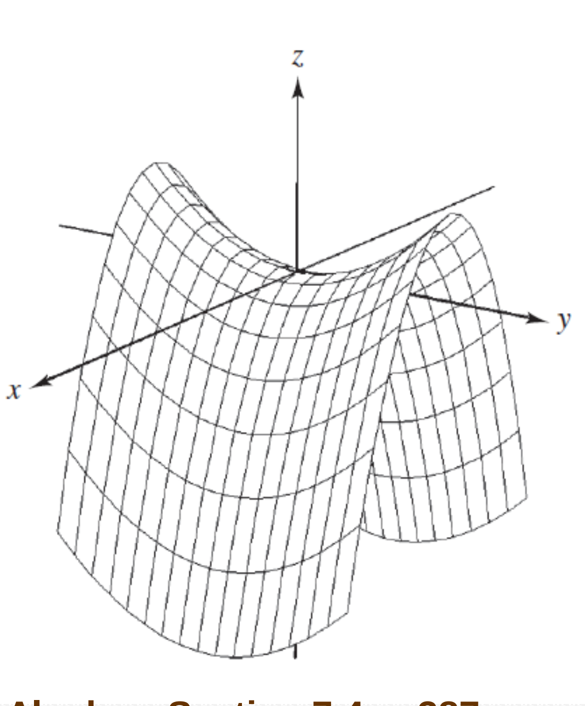


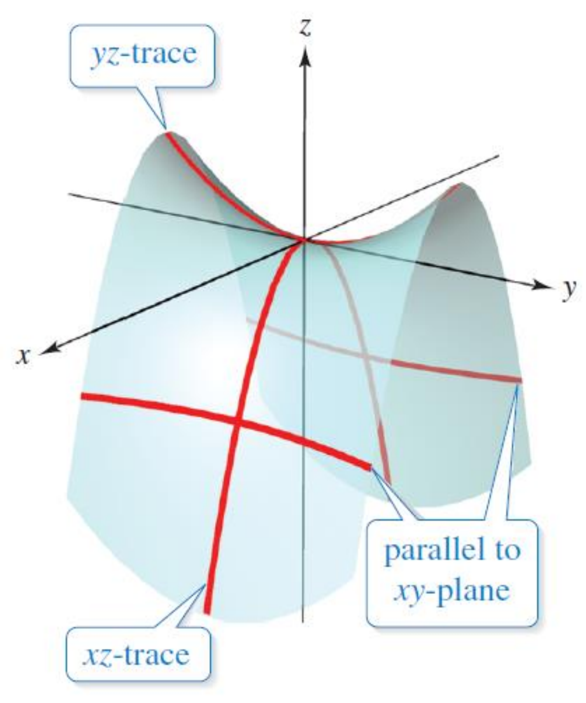
- Hyperbolic Paraboloid

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

Trace	Plane
Hyperbola	Parallel to xy-plane
Parabola	Parallel to xz-plane
Parabola	Parallel to yz-plane

The axis of the paraboloid corresponds to the variable raised to the first power.





- Ex 8: (Rotation of a Quadric Surface)

$$5x^2 + 4y^2 + 5z^2 + 8xz - 36 = 0$$

Sol:

$$A = \begin{bmatrix} 5 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 5 \end{bmatrix}$$

$$\lambda_1 = 1, \lambda_2 = 4, \text{ and } \lambda_3 = 9$$

$$(x')^2 + 4(y')^2 + 9(z')^2 - 36 = 0$$

$$\frac{(x')^2}{6^2} + \frac{(y')^2}{3^2} + \frac{(z')^2}{2^2} = 1$$

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Key Learning in Section 7.4

- Model population growth using an age transition matrix and an age distribution vector, and find a stable age distribution vector.
- Use a matrix equation to solve a system of first-order linear differential equations.
- Find the matrix of a quadratic form and use the Principal Axes Theorem to perform a rotation of axes for a conic and a quadric surface.
- Solve a constrained optimization problem.

Keywords in Section 7.4

- population growth: 人口成長
- age distribution vector: 年齡分佈向量
- age transition matrix: 年齡轉換矩陣
- quadratic form: 二次式
- quadratic equation: 二次方程式
- principal axes theorem: 主軸定理

- Diffusion

Eigenvalues and eigenvectors are useful for modeling real-life phenomena. For example, consider an experiment to determine the diffusion of a fluid from one flask to another through a permeable membrane and then out of the second flask, researchers determine that the flow rate between flasks is twice the volume of fluid in the first flask and the flow rate out of the second flask is three times the volume of fluid in the second flask, then the system of linear differential equations below, where y_i represents the volume of fluid in flask i, models this situation.

$$y_1' = -2y_1$$
$$y_2' = 2y_1 - 3y_2$$

In Section 7.4, you will use eigenvalues and eigenvectors to solve such systems of linear differential equations. For now, verify that the solution of this system is

$$y_1 = C_1 e^{-2t}$$
$$y_2 = 2C_1 e^{-2t} + C_2 e^{-3t}.$$

Genetics

Genetics is the science of heredity. A mixture of chemistry and biology, genetics attempts to explain hereditary evolution and gene movement between generations based on the deoxyribonucleic acid (DNA) of a species. Research in the area of genetics called population genetics, which focuses on genetic structures of specific populations, is especially popular today. Such research has led to a better understanding of the types of genetic inheritance. For instance, in humans, one type of genetic inheritance is called *X–linked inheritance* (or *sex-linked inheritance*), which refers to recessive genes on the X chromosome. Males have one X and one Y chromosome, and females have two X chromosomes. If a male has a defective gene on the X chromosome, then its corresponding trait will be expressed because there is not a normal gene on the Y chromosome to suppress its activity. With females, the trait will not be expressed unless it is present on both X chromosomes, which is rare. This is why inherited diseases or conditions are usually found in males, hence the term sex-linked inheritance. Some of these include hemophilia A, Duchenne muscular dystrophy, red-green color blindness, and hereditary baldness. Matrix eigenvalues and diagonalization can be useful for coming up with mathematical models to describe X-linked inheritance in a given population.

- Relative Maxima and Minima

The *Hessian matrix* is a symmetric matrix that can be helpful in finding relative maxima and minima of functions of several variables. For a function f of two variables x and y—that is, a surface in R^3 —the Hessian matrix has the form

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}.$$

The determinant of this matrix, evaluated at a point for which f_x and f_y are zero, is the expression used in the Second Partials Test for relative extrema.

- Architecture

Some of the world's most unusual architecture makes use of quadric surfaces. For example, *Catedral Metropolitana Nossa Senhora Aparecida*, a cathedral located in Brasilia, Brazil, is in the shape of a hyperboloid of one sheet. It was designed by Pritzker Prize winning architect Oscar Niemeyer, and dedicated in 1970. The sixteen identical curved steel columns are intended to represent two hands reaching up to the sky. In the triangular gaps formed by the columns, semitransparent stained glass allows light inside for nearly the entire height of the columns.